NONLINEAR MAPPING OF UNCERTAINTIES: A DIFFERENTIAL ALGEBRAIC APPROACH

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ABSTRACT

A method for the nonlinear propagation of uncertainties in celestial mechanics based on differential algebra is presented. The arbitrary order Taylor expansion of the flow of ordinary differential equations with respect to the initial condition delivered by differential algebra is exploited to implement an accurate and computationally efficient Monte Carlo algorithm, in which thousands of pointwise integrations are substituted by polynomial evaluations. The algorithm is applied to study the close encounter of asteroid Apophis with our planet in 2029. To this aim, we first compute the high order Taylor expansion of Apophis' close encounter distance from the Earth by means of map inversion and composition; then we run the proposed Monte Carlo algorithm to perform the statistical analysis.

Key words: Uncertainties Propagation; Monte Carlo Simulation; Apophis Close Encounter; Differential Algebra.

1. INTRODUCTION

The propagation of uncertainties in orbital mechanics is usually addressed by linear propagation models [1, 7, 17] or nonlinear Monte Carlo simulations [14]. The main advantage of the linear methods is the simplification of the problem, but their accuracy drops off for highly nonlinear systems and/or long time propagations. On the other hand, Monte Carlo simulations provide true trajectory statistics, but are computationally intensive. The tools currently used for the robust detection and prediction of planetary encounters and potential impacts of Near Earth Objects (NEO) are based on these techinques [8, 9, 15], and thus suffer the same limitations. The effect of the coordinate system on the propagated statistics is analyzed by Junkins et al. [18, 19] and used to develop an alternative approach to orbit uncertainty propagation. However, this method is based on a linear assumption and thus cannot map nonlinearities. An alternate way to analyze trajectory statistics by incorporating higher-order Taylor series terms that describe localized nonlinear motion is proposed by Park and Scheeres [20]. Their approach is based on proving the integral invariance of the probability density function via solutions of the Fokker–Planck equations for diffusionless systems, and by combining this result with the nonlinear state propagation to derive an analytic representation of the nonlinear uncertainty propagation. As a result, the method enables the nonlinear mapping of Gaussian statistics, bypassing the drawbacks of Monte Carlo simulations. However, it is limited to systems derived from a single potential.

Differential algebraic (DA) techniques are proposed as a valuable tool to develop an alternative approach to tackle the previous tasks. Differential algebra supplies the tools to compute the derivatives of functions within a computer environment [4, 5, 6]. More specifically, by substituting the classical implementation of real algebra with the implementation of a new algebra of Taylor polynomials, any function f of n variables is expanded into its Taylor polynomial up to an arbitrary order k. This has an important consequence when the numerical integration of an ordinary differential equation (ODE) is performed by means of an arbitrary integration scheme. Any explicit integration scheme is based on algebraic operations, involving the evaluation of the ODE right hand side at several integration points. Therefore, starting from the DA representation of the initial condition and carrying out all the evaluations in the DA framework, the flow of an ODE is obtained at each step as its Taylor expansion in the initial condition [10]. The availability of such high order expansions is exploited when problems with uncertain initial conditions have to be analyzed. As the accuracy of the Taylor expansion can be kept arbitrarily high by adjusting the expansion order, the approach of classical Monte Carlo simulations can be enhanced by replacing thousands of integrations with evaluations of the Taylor expansion of the flow. As a result, the computational time reduces considerably without any significant loss in accuracy.

The algorithm is applied to the prediction of Apophis planetary encounter and potential impact, taking into account its measurement uncertainties. The availability of high order maps in space and time, and intrinsic tools for their inversion, are exploited to reduce the computation of the close encounter distance (CED) from the Earth of all the asteroids belonging to the initial uncertainty cloud (commonly referred to as virtual asteroids [16]) to the simple evaluation of polynomials. Similar techniques exploiting high order Taylor expansions of the flow of ODE and their inverses obtained with DA techniques have already been efficiently utilized in beam physics. Two noticeable applications are the reconstruction of trajectories in particle spectrographs together with the reconstructive correction of residual aberrations [2], and the end-to-end simulations of fragment separators [11]. This paper presents an application to celestial mechanics.

2. DA INTEGRATION OF APOPHIS DYNAMICS

2.1. Dynamical Models

The study of the motion of a NEO in the Solar System with an accuracy sufficient to predict collisions requires the inclusion of various relativistic corrections to the well-known Newtonian forces based on the Kepler's force law. Specifically, the full equation of motion in the Solar System including the relevant relativistic effects are considered.

It is assumed that the object we are integrating is affected by the gravitational attraction of k bodies, but has no gravitational effect on them; i.e., we are adopting the restricted $(n+1)$-body problem approximation. The positions, velocities, and accelerations of the k bodies are considered as given values, computed by cubic spline interpolations of data retrieved from HORIZONS Web-Interface (http://ssd.jpl.nasa.gov/horizons.cgi). These interpolations are necessary as in the DA framework the use of external code is not permitted. The cubic splines are built so as to keep the maximum error with respect to HORIZONS' ephemerides of the order of 10^{-9} AU and 10^{-10} AU/day for bodies' position and velocity, respectively (see [3] for details). In our integrations k includes the Sun, planets, the Moon, Ceres, Pallas, and Vesta. For planets with moons, with the exception of the Earth, the center of mass of the system is considered. The dynamical model is written in the J2000.0 Ecliptic reference frame and is commonly referred to as Standard Dynamical Model [12]. To improve the integration accuracy the dynamics are scaled by Earth semimajor axis and Sun gravitational parameter (i.e., $a_E = 1$ and $\mu_S = Gm_s = 1$). We must mention that, to obtain a full understanding of the dynamics of a body in the Solar System, other effects should be taken into account, such as: the forces due to other natural satellites and asteroids, the J_2 (and higher order harmonics of the potential) effect of the Earth and other bodies, Yarkosvsky and solar radiation pressure effects [12].

When the asteroid approaches the Earth, a different set of ODE are integrated to avoid cancellation errors associated to repetitive subtraction of Apophis' and Earth's position vectors occurring across the flyby pericenter. In this case the equations of motion are written in the J2000.0 Earth-Centered Inertial reference frame. The same gravitational bodies of the heliocentric phase are considered, whereas relativistic corrections are neglected as their effect during a fast close encounter is negligible. In this phase the dynamics are scaled by the radius of the Earth and by the Earth gravitational parameter (i.e., $R_E = 1$ and $\mu_E = Gm_E = 1$).

2.2. Flow Expansion

The high order expansion of the flow of ODE can be straightforwardly obtained by evaluating any explicit numerical integration scheme within the DA framework. The results presented here are obtained by applying a DA-based 8-th order Runge–Kutta–Fehlberg (RKF78) scheme with absolute and relative tolerance of 10^{-12}. The integration window is June 18, 2009 to April 16, 2029, being April 13, 2029, the day in which the close approach occurs.

The nominal initial state and the associated σ of Apophis, expressed in equinoctial variables $\boldsymbol{p} = (a, P_1, P_2, Q_1, Q_2, l)$, are taken from the Near Earth Object Dynamic Site (newton.dm.unipi.it/neodys) and summarized in Table 1. Apophis' initial condition is initialized

Table 1. Apophis equinoctial variable at 3456 MJD2000 (June 18, 2009) and associated σ values. Units: a [AU] and l [deg].

	Nom Value	σ
a	0.922438242375914	2.29775×10^{-8}
P_1	-0.093144699837425	3.26033×10^{-8}
P_2	0.166982492089134	7.05132×10^{-8}
Q_1	-0.012032857685451	5.39528×10^{-8}
Q_2	-0.026474053361345	1.83533×10^{-8}
l	88.3150906433494	6.39035×10^{-5}

as DA variables $[\boldsymbol{p}_0] = \boldsymbol{p}_0 + 3\boldsymbol{\sigma}\delta\boldsymbol{p}_0$, where $3\boldsymbol{\sigma}$ is used as a scaling factor. These variables are converted into cartesian coordinates using the relations given in Battin [1], evaluated in the DA framework and then numerically propagated. Note that the solution of the Kepler equation, required for the computation of the eccentric longitude, is carried out by applying the DA-algorithm introduced in Di Lizia et al. [10].

The nominal heliocentric trajectories of Apophis and the Earth are shown in Fig. 1 by the solid and dotted lines, respectively.

Figure 2 shows a zoom of Apophis' close approach with the Earth in the geocentric reference frame. It is worth mentioning that the maximum norm of the difference between the computed trajectory and Apophis' HORIZONS ephemerides

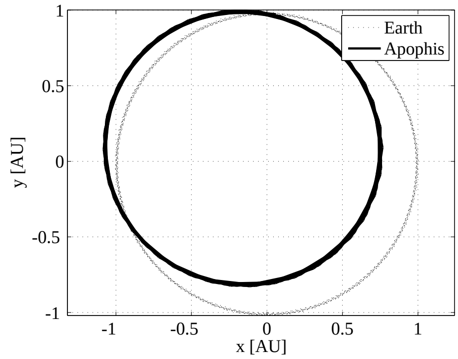

Figure 1. Apophis heliocentric phase trajectory.

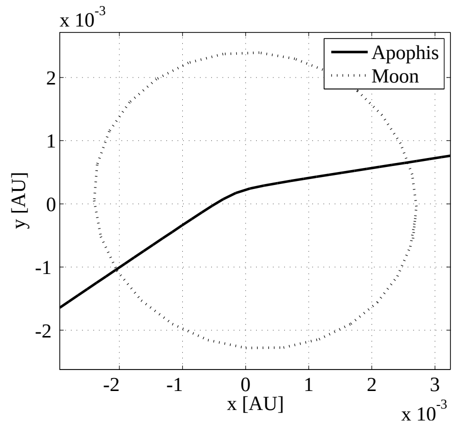

Figure 2. Apophis close encounter trajectory.

is less than 5×10^{-8} AU. The mismatch is due to all different initial conditions, dynamical and ephemeris model, and integration scheme.

An analysis on the accuracy of the flow expansion is mandatory before introducing the DA-based Monte Carlo algorithm. Figures 3 and 4 show the maximum position and velocity error of the Taylor representation of the flow at the corners of the initial set, with respect to the pointwise integration of the same points. Initial widths of 3, 6, and 9 σ and expansion orders from 1 to 5 are considered. The expansion error decreases when higher expansion orders are selected and when smaller uncertain sets are considered. The errors tend to decrease exponentially with the expansion order, until reaching a lower limit of approximately 5×10^{-11} [AU] on position and 3×10^{-10} [AU/day] on velocity. It is worth notic-

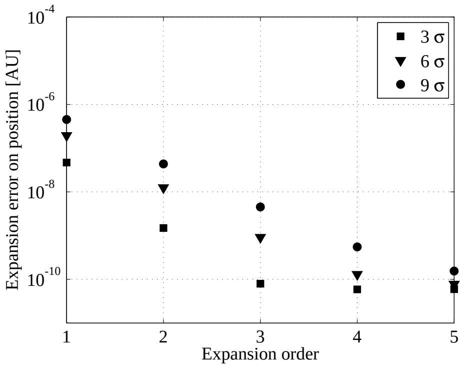

Figure 3. Accuracy of the Taylor expansion of the flow corresponding to different expansion orders and initial uncertainty sets: position error.

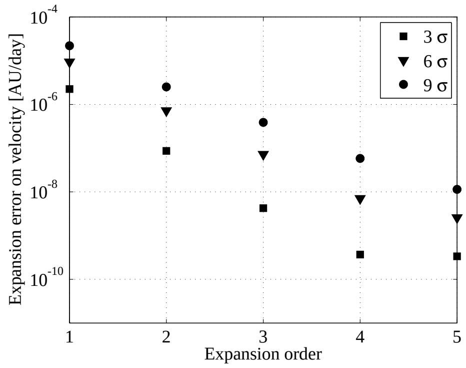

Figure 4. Accuracy of the Taylor expansion of the flow corresponding to different expansion orders uncertainty sets: velocity error.

ing that a fifth order expansion guarantees a gain of approximately three order of magnitude in the flow representation with respect to linear methods. This gain can be crucial when impact probability and/or resonant returns are studied. The figure clearly shows that Taylor polynomial accuracy is a function of both the expansion order and domain width. The drawback for obtaining the Taylor expansion of the flow with respect to the initial condition is the computational time to perform a single integration, as shown in Fig. 5. In this figure the ratio between the computational time of a k-th order DA integration and a single pointwise integration is illustrated, underlining that a 5-th order integration is approximately eight times slower. On the other hand, the availability of the flow expansion enables the development of a computationally efficient Monte Carlo method, as described in the next section.

3. DA-BASED MONTE CARLO

Within the dynamical models adopted and the chosen integration scheme, the asteroid reference solution has a close encounter distance from the Earth center of mass of 38161.55420 km at epoch 10695.907094 MJD2000. In order to evaluate the possibility of an Earth's impact it is necessary to accurately propagate the statistics of the asteroid. The accurate computation of statistics in nonlinear dynamical systems often relies on Monte Carlo simulations. The algorithmic flow of a Monte Carlo simulation is:

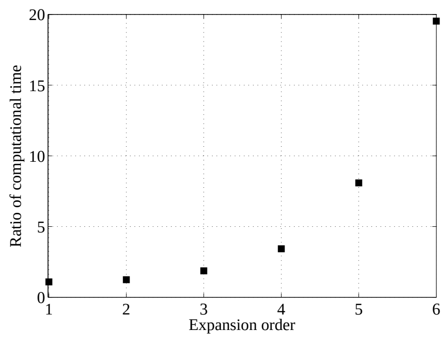

Figure 5. DA integration computational time compared to single pointwise integration.

1. Generate random samples based on the statistical distribution of the uncertainty to be propagated.
2. Run a pointwise integration of each sample in the fully nonlinear dynamics.
3. Perform the statistical analysis of the results.

There are three critical disadvantages when using this approach:

- convergence of the statistics usually requires a large number of sample trajectories to be propagated,

- the simulation needs to be repeated for different initial distributions,
- it does not provide the user with analytical information, useful for additional analyses.

These problems affect both the computational burden associated to the Monte Carlo simulation and its validity for different statistics [20]. The previous drawbacks become dramatic when thousands of long-term integrations are required, as for the analysis of possible NEO close encounter with the Earth [16].

DA integration delivers an arbitrary order Taylor expansion of the flow of the ODE, which is analytic. Furthermore, the accuracy of the map expansion can be controlled by acting on the expansion order. For these reasons, it is possible to substitute the thousands of pointwise integrations required for classical Monte Carlo simulations with an equal number of map evaluations, i.e. fast polynomials evaluations.

The resulting DA-based Monte Carlo simulation can be summarized as:

1. Perform a single DA integration selecting the expansion order according to the demanded accuracy.
2. Generate random samples based on the statistical distribution of the uncertainty to be propagated.
3. Evaluate the flow expansion map for all the samples, requiring only fast polynomial evaluations.
4. Perform the statistical analysis of the results.

The ratio between the computational time of a DA-based Monte Carlo simulation and its pointwise counterpart is given by

$$\frac{t_n + n_s t_e}{n_s t_0}, \tag{1}$$

where t_n, t_e, and t_0 are the computational times of a k-th order DA integration, a flow map evaluation, and a pointwise integration, respectively; and n_s the number of samples of the Monte Carlo simulation. The computational cost of a Taylor map evaluation, depends on the expansion order, but it is negligible compared to a pointwise integration. For this reason, expression (1) can be approximated by $\frac{m}{n_s}$, in which m is the ratio between the computational time of a k-th order DA integration and a pointwise integration (see Fig. 5). The value of m strongly depends on the expansion order, but it is few orders of magnitude smaller than the number of samples required for a good representation of the statistics. For this reason, the ratio $\frac{m}{n_s}$ is small, proving that the proposed DA-based Monte Carlo simulation is computationally efficient. As an example, in Sec. 5, Fig. 10 will show that the computational time is reduced by a factor of at least 100 for a typical sample size of 10000 virtual asteroids.

In case new statistics need to be propagated, it is not necessary to perform an additional DA integration as only steps 2–4 are required. Furthermore, if the statistical analysis is performed for a different final time, the possibility of obtaining Taylor expansions with respect to the final time can be exploited (see Sec. 6.1). Moreover, as the flow expansion is analytical, an analytic framework is delivered. In conclusion, all the major drawbacks of a classical Monte Carlo approach are circumvented. These properties are better highlighted in Sect. 6 by applying the algorithm to the study of Apophis' close encounter with the Earth in 2029.

4. CED ALGORITHM

Let us suppose the close approach of the nominal asteroid occurs at the epoch t_f, and consider the integration of the asteroid dynamics from $t = t_0$ to $t = t_f$. Initialize the initial state and the final integration epoch as DA variables; i.e.,

$$\begin{aligned} [\boldsymbol{x}_0] &= \boldsymbol{x}_0 + \delta\boldsymbol{x}_0 \\ [t_f] &= t_f + \delta t_f, \end{aligned} \tag{2}$$

where $\boldsymbol{x}_0$ is the initial condition corresponding to the nominal asteroid. Using the DA-based RKF78 integrator obtain the map

$$[\boldsymbol{x_f}] = \boldsymbol{x}_f + \mathcal{M}_{\boldsymbol{x}_f}(\delta\boldsymbol{x}_0, \delta t_f). \tag{3}$$

The map (3) is the k-th order Taylor expansion of the flow with respect to the initial condition and the final epoch about their nominal values $\boldsymbol{x}_0$ and t_f. Based on a mere DA-based computation, the final solution $\boldsymbol{x}_f$ can be used to compute the Taylor expansion of distance from the Earth

$$[d_f] = d_f + \mathcal{M}_{d_f}(\delta\boldsymbol{x}_0, \delta t_f). \tag{4}$$

More specifically, map (4) describes how the distance varies depending on the virtual asteroid and the final integration epoch.

Using the derivation operator available in the DA framework, the Taylor expansion of the derivative $d'_f = \mathrm{d}(d_f)/\mathrm{d}t_f$ can be obtained

$$[d'_f] = d'_f + \mathcal{M}_{d'_f}(\delta\boldsymbol{x}_0, \delta t_f). \tag{5}$$

The constant part of the map (5), d'_f, is the derivative of the distance from the Earth of the nominal solution at its close encounter; i.e., at CED epoch. Consequently, this is a stationary point for the nominal solution, and $d'_f = 0$. Then, the map (5) reduces to

$$\delta d'_f = \mathcal{M}_{d'_f}(\delta\boldsymbol{x}_0, \delta t_f), \tag{6}$$

in which we omit the bracket operator for the sake of a simpler notation. Consider the map

$$\begin{pmatrix} \delta d'_f \\ \delta\boldsymbol{x}_0 \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{d'_f} \\ \mathcal{I}_{\boldsymbol{x}_0} \end{pmatrix} \begin{pmatrix} \delta\boldsymbol{x}_0 \\ \delta t_f \end{pmatrix}, \tag{7}$$

which is built by concatenating $\mathcal{M}_{d'_f}$ with the identity map for δx_0. Map (7) can now be inverted to obtain

$$\begin{pmatrix} \delta\boldsymbol{x}_0 \\ \delta t_f \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{d'_f} \\ \mathcal{I}_{\boldsymbol{x}_0} \end{pmatrix}^{-1} \begin{pmatrix} \delta d'_f \\ \delta\boldsymbol{x}_0 \end{pmatrix}. \tag{8}$$

This is a full nonlinear map inversion that is obtained by applying the algorithm illustrated in [5]. This algorithm reduces the map inversion problem to the solution of an equivalent fixed point problem, which can be solved with a fixed amount of effort in the DA setting.

Map (4) is then concatenated to the identity map for δt_f to obtain

$$\begin{pmatrix} d_f \\ \delta t_f \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{d_f} \\ \mathcal{I}_{t_f} \end{pmatrix} \begin{pmatrix} \delta\boldsymbol{x}_0 \\ \delta t_f \end{pmatrix}. \tag{9}$$

Map (9) can now be composed with map (8) to obtain

$$\begin{pmatrix} d_f \\ \delta t_f \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{d_f} \\ \mathcal{I}_{t_f} \end{pmatrix} \circ \begin{pmatrix} \mathcal{M}_{d'_f} \\ \mathcal{I}_{\boldsymbol{x}_0} \end{pmatrix}^{-1} \begin{pmatrix} \delta d'_f \\ \delta\boldsymbol{x}_0 \end{pmatrix}, \tag{10}$$

which relates d_f and δt_f to the displacements of the derivative of the final distance $\delta d'_f$ and of the state vector of the virtual asteroid $\delta\boldsymbol{x}_i$ from their values. As for the reference value $d'_f = 0$, the necessary condition for CED computation is

$$\delta d'_f = 0. \tag{11}$$

Substituting into (10) yields

$$\begin{pmatrix} {d_f}^* \\ \delta {t_f}^* \end{pmatrix} = \begin{pmatrix} \mathcal{M}_{d_f} \\ \mathcal{I}_{t_f} \end{pmatrix} \circ \begin{pmatrix} \mathcal{M}_{d'_f} \\ \mathcal{I}_{\boldsymbol{x}_0} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \delta\boldsymbol{x}_0 \end{pmatrix}. \tag{12}$$

Eventually, map (12) delivers the desired explicit relation between the CED (${d_f}^*$) and the epoch at which it is reached ($t_f + \delta {t_f}^*$) with the displacement $\delta\boldsymbol{x}_i$ in terms of Taylor polynomials. Given any virtual asteroid belonging to the initial set (which corresponds to a specific value of the displacement $\delta\boldsymbol{x}_i$), the simple evaluation of the polynomials in (12) delivers the CED and the epoch at which it is reached. In this way, the problem highlighted by Milani et al. [15] is solved.

5. CED STATISTICAL ANALYSIS

The DA-based Monte Carlo simulation introduced in Sec. 3 is run on map (12) to perform the nonlinear mapping of the initial uncertainties on the CED. More specifically, 10000 virtual asteroids are generated with a normal random distribution with mean value and standard deviation as in Table 1. For each sample, the displacement with respect to the nominal initial conditions is computed and map (12) is evaluated to obtain its CED and the associated epoch. The result is reported in Fig. 6 in terms of probability distribution for the CED. The analysis of the results shows that the mean value is 38161.54 km with a standard deviation of 492.1 km, thus the possibility of having an Earth impact in 2029 is ruled out.

For the same virtual asteroids, map (12) is also evaluated to obtain the close encounter epochs. The result is presented in Fig. 7 in terms of the probability distribution of the displacement $\delta {t_f}^*$ from the nominal epoch t_f. The maximum displacement is of the order of 30 s. In Fig. 8 the

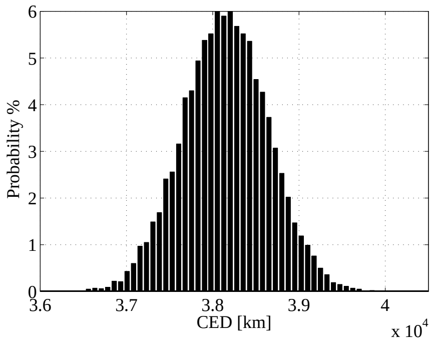

Figure 6. Monte Carlo analysis of virtual asteroids CED.

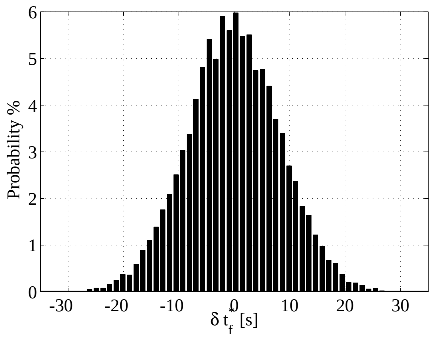

Figure 7. Monte Carlo analysis of virtual asteroids CED epochs.

10000 virtual asteroids are plotted in the CED-δt_f^* plane.

For the sake of completeness, an accuracy analysis of the results is presented in Fig. 9. Ten virtual asteroids are randomly selected from the initial set. For each virtual asteroid, the minimum distance and the corresponding epoch, resulting from map (12), are reported in the figure. Then, a pointwise integration of the motion of each asteroid is performed to obtain the profile of Earth's distances shown in the dotted lines. Although the accuracy on the identification of the epoch of the close encounter is not clearly visualized, due to the very little displacement in $t_f{}^*$, it is clearly shown that the algorithm is able to accurately identify the CED values of the resulting trajectories. Figure 10 concludes the analysis by showing the ratio of the computational time between the proposed DA-based Monte Carlo simulation and its pointwise counterpart as a function of the expansion order when 10000 virtual asteroids are considered. It is apparent that the drawback of the higher computational cost required by a DA integration is rewarded by the significant time saving achieved by substituting 10000 pointwise integrations with the same number of polynomials evaluations.

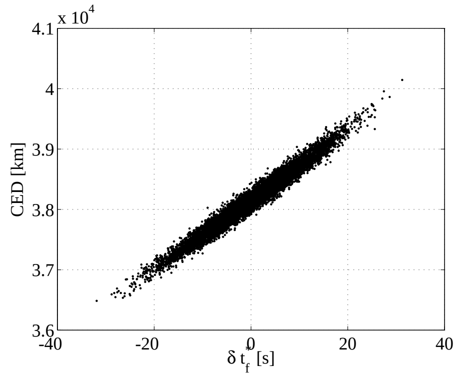

Figure 8. CED vs δt_f^* for 10000 virtual asteroids.

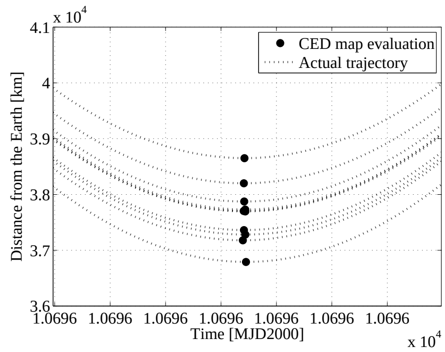

Figure 9. Accuracy of the CED algorithm: virtual asteroids actual trajectories.

6. CONCLUSIONS

The paper introduced a Monte Carlo simulation based on the high order Taylor expansion of the flow of ODE, enabled by the use of differential algebra. Being based on the replacement of pointwise integrations with fast evaluation of polynomials, the proposed algorithm guarantees significant computational time savings. The ac-

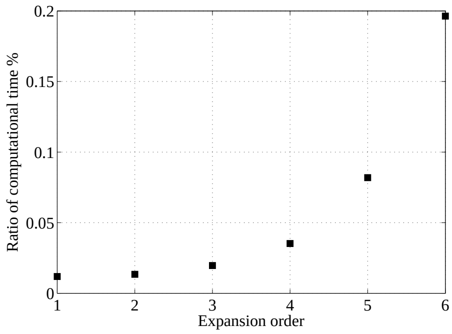

Figure 10. Percentage of computational time required by a DA-based Monte Carlo run versus a classical Monte Carlo simulation for 10000 virtual asteroids.

curacy of the algorithm can be suitably tuned by varying the flow expansion order. Furthermore, the availability of analytic Taylor expansions and the use of DA embedded tools as map inversion, composition, and derivation allows the user to compute arbitrary order maps of the quantities on which the statistical analysis is performed; thus, the algorithm is not limited to the flow of ODE. More specifically, a technique for the automatic computation of both CED and CED epochs for all the virtual asteroids belonging to the initial uncertainty cloud has been developed. The efficiency and effectiveness of the methods are proven by applying them to the analysis of Apophis close encounter with the Earth occurring in April 2029. In particular, it is shown that

- the nonlinear mapping of uncertainties can be performed for any complex and arbitrary dynamics, even when long-term integrations are required;
- a fifth order expansion increases the accuracy of the computation of the CED by approximately two orders of magnitude with respect to classical linear methods;
- the expansion in time allows for the proper identification of the CED epoch for all the virtual asteroids.

As an additional result, the occurrence of an impact with the Earth in April 2029 can be ruled out.

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REFERENCES

[1] Battin, R.H.: An Introduction to the Mathematics and Methods of Astrodynamics. AIAA Education Series, New York, 1968.

[2] Berz, M., Joh, K., Nolen, J.A., Sherrill, B.M., and Zeller, A.F.: Reconstructive correction of aberration in nuclear particle spectrographs. Physical Review C, **47**, pp 537-544, (1993).

[3] Bernelli-Zazzera, F., Berz, M., Lavagna, M., Makino, K., Armellin, R., Di Lizia, P., Jagasia, R., and Topputo, F.: NEO Encounter 2029: Orbital Prediction via Differential Algebra and Taylor Models. Final Report, Ariadna id: 08-4303, Contract No. 20271/06/NL/HI (2009).

[4] Berz, M.: Differential Algebraic Techniques. Entry in Handbook of Accelerator Physics and Engineering, World Scientific, New York, (1999a).

[5] Berz, M.: Modern Map Methods in Particle Beam Physics. Academic Press, (1999b).

[6] Berz, M., and Makino, K.: COSY INFINITY version 9 reference manual. MSU Report MSUHEP-060803, Michigan State University, East Lansing, MI 48824, pp 1-84, (2006).

[7] Crassidis, J.L., and Junkins, J.L.: Optimal Estimation of Dynamics Systems. CRC Press LLC, Boca Raton, FL, pp 243-410, (2004).

[8] Chesley, S.R., and Milani, A.: An automatic Earth-asteroid collision monitoring system. Bull. Am. Astron. Soc., **32**, pp 682, (2000).

[9] Chodas, P.W., and Yeomans, D.K.: Predicting close approaches and estimating impact probabilities for near-Earth projects. AAS/AIAA Astrodynamics Specialists Conference, Girdwood, Alaska, (1999).

[10] Di Lizia, P.: Robust Space Trajectory and Space System Design using Differential Algebra. Ph.D. dissertation, Politecnico di Milano (2008).

[11] Erdelyi, B., Bandura, L., Nolen, J., and Manikonda, S.: Code development for Next-Generation High-Intensity Large Acceptance Fragment Separators, Proceedings of PAC07, Albuquerque, New Mexico, USA, (2007).

[12] Giorgini, J.D., Benner, L.A.M., Ostro, S.J., Nolan, M.C., and Busch, M.W.: Predicting the Earth encounters of (99942) Apophis, Icarus 193, pp 1–19, (2008).

[13] Hoefkens, J., Berz, M., and Makino, K.: Controlling the Wrapping Effect in the Solution of ODEs for Asteroids, Reliable Computing, **8**, pp 21-41, (2003).

[14] Maybeck, P.S., Stochastic Models, Estimation, and Control. Academic Press, New York, **2**, pp 159–271, (1982).

[15] Milani, A., Chesley, S.R., and Valsecchi, G.B.: Asteroid close encounters with Earth: Risk assessment. Planet. Space Sci., **48**, pp 945–954, (2000).

[16] Milani, A. and Chesley, S.R. and Chodas, P.W. and Valsecchi, G.B.: Asteroid close approaches: analysis and potential impact detection. Asteroids III, pp 89–101, (2002).

[17] Montenbruck, O., and Gill, E.: Satellite Orbits. 2nd ed., SpringerVerlag, New York, pp 257–291, (2001).

[18] Junkins, J., Akella, M., and Alfriend, K.: Non-Gaussian Error Propagation in Orbit Mechanics. Journal of the Astronautical Sciences, **44**, pp 541–563, (1996).

[19] Junkins, J., and Singla, P.: How Nonlinear Is It? A Tutorial on Nonlinearity of Orbit and Attitude Dynamics. Journal of the Astronautical Sciences, **52**, pp 7–60 (2004).

[20] Park, R., and Scheeres, D.: Nonlinear Mapping of Gaussian Statistics: Theory and Applications to Spacecraft Trajectory Design. Journal of guidance, Control, and Dynamics, **29**, pp 1367–1375, (2006).

[21] Park, R.S. and Scheeres, D.J.: Nonlinear Semi-Analytic Methods for Trajectory Estimation. Journal of Guidance, Control and Dynamics, **30**, 1668–1676, (2007).

[22] Seidelmann, P.K.: Explanatory Supplement to the Astronomical Almanac. University Science Books, Mill Valley, California, (1992).