Applications of Higher Order Convergence Forms in Interval Global Optimization Techniques

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1 The Global Optimization Problem

- Let $\Re$ be the set of reals, $\mathbf{X} \subseteq \Re^l$ be a box.
- Let
$$f : \mathbf{X} \rightarrow \Re$$
be a $m+1$ times differentiable function for some positive integer m.
- Let $\bar{f}(\mathbf{X})$ denote the set of all values of f on $\mathbf{X}$.
- **Problem**: To efficiently determine arbitrarily good lower bounds for the minimum of $\bar{f}(\mathbf{X})$.

2 Interval Analysis Approach to Global Optimization

- Many IA algorithms for solving this global optimization problem.
- IA methods use branch and bound techniques.
- That is,
  - Start from initial box $\mathbf{X}$,
  - Subdivide $\mathbf{X}$ and store the subboxes in a list,
  - Discard subboxes which are guaranteed not to contain a global minimizer.
  - Repeat until desired accuracy in terms of the width of intervals in list is achieved.
- A basic branch and bound algorithm of IA is Moore-Skelboe (MS) algorithm.
- The MS algorithm is reliable.
- **Problem** with MS Algorithm:
  - Slow to converge in 'difficult' problems.
  - The convergence is linear.

3 Motivation

- MS algorithm is slower for inclusion functions of first and sometimes even second orders.
- Faster convergence possible with higher order inclusion function forms for ’difficult’ problems.
- So higher order inclusion forms for multidimensional functions are needed to sought.

4 Objective

- To develop unconstrained global optimization algorithm(s) with the developed higher order inclusion function form, for efficient determination of arbitrarily good lower bounds on the minimum of $\bar{f}(\mathbf{X})$.
- In each case, the practical effectiveness of the proposed tool is to be numerically tested and compared with existing techniques on several 'difficult' problems of different dimensions.

5 Our strategy

- Get Taylor form of the objective function f.
- Use Bernstein polynomials for bounding polynomial part.
- Resulting form is Taylor-Bernstein (TB) form.
- TB form exhibits high order convergence.
- Then use TB form as an inclusion function of f in MS algorithm.
- Form combined inclusion form using TB and LR forms.
- Use several inclusion function forms in Global Optimization algorithm.

6 Main contributions

A.

- We first propose an improved Taylor-Bernstein (TB) form (compared to that of Lin and Rokne, 1996) as a higher order inclusion function form for multidimensional functions.
- This improved TB form uses Bernstein polynomials for bounding the range of the polynomial obtained from Taylor form of function f.
- The higher order convergence behavior of improved TB is form is tested against Lin Rokne's form (LR) on six benchmark examples of dimensions one to six.
- Unlike with LR form higher order convergence upto order 9 is obtained in all examples.

B.

- An algorithm for unconstrained global optimization is proposed in the framework of the MS algorithm.
- The proposed algorithm uses improved TB form as an inclusion function.
- We also modify cutoff test and termination condition in the algorithm.

- The performance of the algorithm is tested on six benchmark examples of varying dimensions
- The proposed algorithm with Taylor order 4 is found to be the best in terms of computational time, space complexity and number of iterations.

C.

- A Combined TB form(CTB) is proposed as a higher order inclusion function form for multidimensional functions.
- In application problems where the domain shrinks from large to small widths, the CTB form found to be more effective than either LR form or the improved TB form.
- The higher order convergence of CTB is tested against LR form and improved TB form on six benchmark examples for very high domains widths to very small domain widths.
- In all examples unlike LR and improved TB form, CTB form succeeds in computing the range enclosures for any dimension and for any domain width.
- Also the CTB form is better in terms of computational time and space complexity than improved TB form for large to intermediate domain widths.

D.

- An improved algorithm for unconstrained global optimization is proposed in the framework of the MS algorithm.
- A novel and powerful feature of this is that, a variety of inclusion function forms for the objective functions are incorporated into it - the CTB form(which in turn includes TB and LR forms), Taylor model and simple natural inclusion form.
- Several improvements are incorporated in Bernstein steps like efficient bisection direction selection, monotonicity test, and cutoff test.
- Also several inclusion functions allows even better cutoff and termination conditions in the global optimization algorithm.

7 Ideas for a New Optimization Algorithm

- Consider a leading box $\mathbf{Y}$ in a given iteration of the MS algorithm.
- Apply Algorithm TB to get an enclosure of $\overline{f}(\mathbf{Y})$ using $F_{TB}(\mathbf{Y})$.
- Then,

$$\min \bar{p}(\mathbf{Y}) + \min R(\mathbf{Y}) \leq \min \overline{f}(\mathbf{Y}) \leq \min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})$$

- Therefore, redefine cut-off level in the MS algorithm as $z = \min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})$
- This is smaller and hence more effective than the original cut-off level $z = \max F(\mathbf{Y}) = \max \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})$.
- Also, error on $\min \overline{f}(\mathbf{Y})$ is no greater than $\max R(\mathbf{Y}) - \min R(\mathbf{Y}) = w(R(\mathbf{Y}))$
  - So, redefine the termination condition in MS algorithm based on the width of $R(\mathbf{Y})$.
  - This is smaller and more effective than the original $w(F(\mathbf{Y})) = w(\bar{p}(\mathbf{Y})) + w(R(\mathbf{Y}))$.
- Based on these ideas, our proposed algorithm uses:
  - The Taylor-Bernstein form F_{TB} as an inclusion function of f.
  - Cut-off value as $z = \min \bar{p}(\mathbf{Y}) + \max R(\mathbf{Y})$ -

compare with the earlier cutoff $z = \max F(\mathbf{Y})$.

– Termination criterion as $w(R(\mathbf{Y}))$ - the earlier criterion used $w(F(\mathbf{Y}))$.

- Finally, our algorithm is called as **T**aylor - **B**ernstein form in **M**oore-**S**kelboe (TBMS) algorithm.

8 Numerical Tests

- Consider the Bard function in More - Garbow-Hillstrom (MGH) test suite.
- For all our computations, we use a PC/Pentium III 800 MHz 256 MB RAM machine with a FORTRAN 90 compiler, and version 8.1 of the COSY-INFINITY package of Berz *et al.*
- The three dimensional function is

$$\begin{aligned} f(x) &= \sum_{i=1}^{15} f_i(x)^2, \\ f_i(x) &= y_i - \left(x_1 + \frac{u_i}{v_i x_2 + w_i x_3} \right), \\ u_i &= i, v_i = 16 - i, w_i = \min(u_i, v_i) \end{aligned}$$

  where, i is given in the paper cited.
- Initial domain is $\left([-0.25, 0.25], [0.01, 2.5]^2\right)$.
- The performances of the various Algorithms are as under.

		TBMS				TMS			
m	ε	It	t	ML	FL	It	t	ML	FL
2	10^{-03}	406	16.64	74	45	3145	76.13	822	772
	10^{-05}	520	32.13	74	7	$*$	$>$ 3600	$*$	$*$
4	10^{-03}	191	35.00	38	7	3124	86.13	818	772
	10^{-05}	202	60.65	38	1	$*$	$>$ 3600	$*$	$*$
6	10^{-03}	162	67.80	38	2	3123	122.81	818	772
	10^{-05}	165	90.22	38	1	$*$	$>$ 3600		
8	10^{-03}	157	79.90	38	2	3122	181.05	818	772
	10^{-05}	159	92.03	38	1	$*$	$>$ 3600	$*$	$*$

MS				
ε	It	t	ML	FL
10^{-03}	6122	466.56	1643	1622
10^{-05}	$*$	$>$ 3600	$*$	$*$

- The global minima found using each of the algorithms is $8.21487....E-03$.

9 Summary of TBMS Performance

- Computational time is quite less in Algorithm TBMS than in MS and TMS for almost all the examples.
- In algorithm TBMS for most of the examples, the computational time first decreases, then increases with m, with the least time required for $m = 4$.
- The speed-up gets better with accuracy.
- In all examples, Algorithm TBMS requires much smaller list lengths and much lesser number of iterations than Algorithm TMS and MS.
- The number of iterations decreases as the order m is increased. A considerable reduction is obtained between $m = 2$ and 4.
- The maximum list length also decreases considerably between $m = 2$ and 4, but decreases little thereafter.
- The best overall choice, in terms of the number of iterations, space-complexity, and speed seems to be Algorithm TBMS with a medium Taylor order $m = 4$.

10 Proposed ‘Combined’ Taylor-Bernstein form

- Improved TB form works quite well compared to Lin-Rokne form for smaller domain widths.
- But it takes long time or even ’fails’ to find the range enclosures for very large domain widths due to excessive Bernstein subdivisions.
- So for the global optimization problems it is required to find the new inclusion function form which works well at all domain widths from very high to very low.
- So we introduced a new form which we call it as Combined TB form or CTB form.
- CTB form uses LR for large domain widths and uses improved TB form for smaller widths.

11 Global optimization using various inclusion function forms

- We proposed higher order convergence forms.
- We also found that natural inclusion form sometimes gives much less overestimation than any of the more sophisticated Taylor or TB forms.
- We now propose Global Optimization algorithm which adaptively switches among various inclusion function forms - F_{NIE} , F_{TM} and F_{CTB} (combination of F_{TB} and F_{LR}).
- We also incorporate several other modifications in Bernstein part.
  (a) A monotonicity test for Bernstein patches similar to that used in MS algorithm.
  (b) A cutoff test for discarding Bernstein patches where surely no global minimizer of p lies.
  (c) An improved strategy for selection of subdivision direction of Bernstein patches.
  (d) We further speed up the Bernstein part by restricting vertex condition for polynomial minima only.
- We also improve cutoff test and termination conditions in the MS algorithm.

12 Numerical Tests

- We solve eleven benchmark problems of various dimensions.
- We compare the performance of various algorithms - MS, MS with Taylor model as inclusion function form, our earlier proposed Algorithm TBMS and CTBMS.
- We compare all these algorithms using the following performance metrics:

1. – Number of algorithmic iterations
   – Computational time
   – Maximum list length
   – Final list length

- We also use the following different evaluation methods to compare algorithms.
  – Ranking
  – statistical measures
  – average and other measures
  – performance profiles

13 Summary of CTBMS Performance

- The proposed algorithm CTBMS is able to solve all the examples while Algorithm MS, Algorithm TMS and Algorithm TBMS could solve about 55%, 45% and 91% of the problems.
- The proposed algorithm CTBMS requires less number of iterations for about 82% of problems.
- It requires less computational time for about 73% of the problems.
- The proposed algorithm requires less list length in about 91% of the problems compared to other algorithms.
- It also requires less final list length in about 73% of problems.

14 Conclusions

- We have successfully devised new inclusion function form(s) for multidimensional functions which gives higher order convergence.
- Using these forms, the algorithm(s) for Global optimization solves 'difficult' problems very well, where traditional algorithm MS fails to solve (converge).
- In each case, the practical effectiveness of the proposed tool is numerically tested and compared with existing techniques on several 'difficult' problems of different dimensions. The results are found to be satisfactory.