A Combined Taylor- Bernstein Form for Higher Order Convergence

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1 Introduction

- An important problem: construction of inclusion functions for multidimensional functions having the property of higher order convergence.
- Such inclusion functions have applications, in the solution of equations, quadrature, global optimization problems, etc.
- In earlier talk, an improved TB form F_{TB} having the property of higher order convergence was presented.
- There, we saw that
  - As domains shrank to small widths, F_{TB} usually successfully computed range enclosure with higher order convergence.
  - But TB form F_{LR} of Lin and Rokne failed to compute range enclosures, due to excessively high degrees of Bernstein form.

- In recent investigations, we found the following.
  - For *large* domain widths, F_{TB} has difficulties in computing range enclosures, due to excessive number of Bernstein subdivisions.
- In many application problems, for example, global optimization problems, we have
  - initial domain widths are quite large,
  - but solution boxes of small widths are required.
- Going by our new findings, neither of existing TB forms may be okay in such problems, over *entire* range of domain widths.
- But, we can have a new inclusion function form that switches between two existing TB forms, depending on domain widths.
- That is, new form behaves as F_{LR} for 'large' domain widths, and as F_{TB} for 'small' domain widths.
- As the domain shrinks from large to small widths, the new TB form is more likely to succeed in computing the range enclosures than existing TB forms.

- In this talk, such a ‘combined’ TB form is described
- It is numerically tested for its effectiveness with those of the existing TB forms, the Taylor model, and the simple natural inclusion form.
- For the testing, six benchmark examples with dimensions varying from 1 to 6 are considered, and the higher order convergence property for orders up to 9 is examined.

2 Taylor form (Recap)

- If $f : \mathbf{X} \to \Re$ is $(m+1)$ times differentiable on $\mathbf{X}$, then Taylor expansion of order m gives

$$f(\mathbf{x}) = \underbrace{f(\mathbf{c}) + \sum_{|\lambda|=1}^{m} \frac{D^{\lambda} f(c)}{\lambda!} (\mathbf{x} - \mathbf{c})^{\lambda}}_{p(\mathbf{x})} + \underbrace{\sum_{|\lambda|=m+1} \frac{f^{(\lambda)}(\boldsymbol{\xi})}{\lambda!} (\mathbf{x} - \mathbf{c})^{m+1}}_{r(\mathbf{x})}, \ \mathbf{x} \in \mathbf{X}, \quad \textbf{(1)}$$

- Here, $\mathbf{c}$ is any point in $\mathbf{X}$ (usually the midpoint), $\boldsymbol{\xi} \in \mathbf{X}$, and

$$\lambda \ : = \{\lambda_1, \ldots, \lambda_l\}, \quad |\lambda| := \lambda_1 + ... + \lambda_l,$$
$$\lambda! \ : = \lambda_1! \ldots \lambda_l!, \quad D^{\lambda} f(x) := \frac{\partial^{\lambda_1 + \ldots + \lambda_l} f(x)}{\partial x_1^{\lambda_1} \ldots \partial x_l^{\lambda_l}}$$

- $p(\mathbf{x})$ is polynomial part and $r(\mathbf{x})$ is remainder part of the Taylor expansion.

- Taylor form F_{Taylor} of order m is

$$F_{Taylor}(\mathbf{X}) := \bar{p}(\mathbf{X}) + R(\mathbf{X})$$

- Taylor form has convergence order $(m+1)$

3 Taylor-Bernstein form of Lin - Rokne (Recap)

- Instead of finding the range $\bar{p}(\mathbf{X})$, Lin and Rokne used Bernstein form for p of degree N'

$$D = (d_1, \ldots, d_l), \quad d_k \geq \frac{1}{w(\mathbf{X})^{m+1}}, \quad k = 1, \ldots, l$$

$$N' = (n'_1, \ldots, n'_l), \quad n'_k := \max\{n_k, d_k\}, \quad k = 1, \ldots, l$$

- They compute a non-sharp enclosure B^* of the range $\bar{p}(\mathbf{X})$ - as the $\min$ to $\max$ over all Bernstein coefficients.

- F_{LR} constructed as

$$F_{LR}(\mathbf{X}) = \mathbf{B}^* + R(\mathbf{X})$$

  retains the $(m+1)$th convergence order property.

- We call the form F_{LR} as the TB form of Lin and Rokne.

- As the domain width $w(\mathbf{X})$ becomes smaller, required degree N' of Bernstein form becomes large, and quite quickly.

- So, as domain intervals shrink in widths, F_{LR}

becomes very computationally intensive

4 Improved Taylor-Bernstein form (Recap)

- In earlier talk, we saw an improved TB form F_{TB} that circumvents the need for high degrees $N' >> N$ of the Bernstein form as for F_{LR}.
- The improved TB form F_{TB} also has the $(m+1)$th convergence order property.
- The step to compute the range $\bar{p}\,(\mathbf{X})$ is based on
  - Bernstein form of minimum required degree N,
  - Subdivision
  - Vertex condition checks.
- The overall method to construct F_{TB} is as follows :

1. Expand the given function f in the Taylor form of order m.
2. Transform the given domain $\mathbf{X}$ to the unit box $\mathbf{U}$, and compute the Bernstein form of degree N.
3. Successively subdivide $\mathbf{U}$ and recompute the Bernstein coefficients on each new subdivision, till the vertex property is satisfied on every subdivision.
4. Compute range $\bar{p}\,(\mathbf{X})$ as minimum to maximum over all Bernstein coefficients.

5. Construct an enclosure for the range of f on X as

$$F_{TB} = \bar{p}\left(\mathbf{X}\right) + R\left(\mathbf{X}\right)$$

5 A new "Combined" TB form

- Typically,
  - F_{TB} requires excessive subdivisions for 'large' $w(\mathbf{X})$,
  - whereas F_{LR} requires excessively high degrees of Bernstein form for 'small' $w(\mathbf{X})$.
- We present a new inclusion form that switches between these two forms depending on domain widths.
- It behaves as F_{LR} for 'large' domain widths, and as F_{TB} for 'small' domain widths.
- For 'large' $w(\mathbf{X})$, we have $D << N$, so $N' = N$. So use F_{LR} based on Bernstein form of degree N
  - rather than use F_{TB} that requires successive subdivisions till the vertex property is attained on every subdivision.
- For 'small' $w(\mathbf{X})$, we have $D >> N$, so $N' >> N$. So, use F_{TB} based on Bernstein form of degree N,
  - rather than F_{LR} that requires Bernstein form of high to very high degree.
- Thus, if $N \geq D$ we use F_{LR}, otherwise we use F_{TB}.
- The resulting form is called the combined TB form F_{CTB}.

- F_{CTB} also has the $(m+1)$th convergence order property.

6 Algorithm for F_{TB}

$[F_{CTB}(\mathbf{X}), \bar{p}(\mathbf{X}), B^*, R(\mathbf{X}), i_f] =$ CTB$(\mathbf{X}, f, m)$

- Note: depending on whether F_{LR} resp. F_{TB} is used, the quantity $\bar{p}(\mathbf{X})$ resp. B^* is set to the empty interval.

1. Expand the given function f in the Taylor form of order m. Use the automated Taylor model technique of Berz *et al.* for this step.
2. Compute the $l-$tuple of indices D given by $D = (d_1, \ldots, d_l)$, where $d_1, \ldots, d_l \geq [1/w(\mathbf{X})]^{m+1}$ If $N \geq D$ then go to the following step, else go to step 7.
3. Set flag $i_f = 0$ and $\bar{p}(\mathbf{X})$ to the empty interval.
4. Transform the given domain $\mathbf{X}$ to the unit box $\mathbf{U}$. Compute Bernstein form of degree N. Compute an enclosure B^* of range $\bar{p}(\mathbf{X})$ as the $\min$ to $\max$ over all Bernstein coefficients.
5. Construct an enclosure for $\bar{f}(\mathbf{X})$ as
$$F_{CTB}(\mathbf{X}) = \mathbf{B}^* + R(\mathbf{X})$$
6. Go to step 11 .
7. Set flag $i_f = 1$ and B^* to empty interval.
8. Transform given domain $\mathbf{X}$ to the unit box $\mathbf{U}$, and

compute Bernstein form of degree N.

9. Successively subdivide $\mathbf{U}$ and recompute Bernstein coefficients on each new subdivision, till vertex property is satisfied on every subdivision.

10. Compute range $\bar{p}(\mathbf{X})$ as minimum to maximum over all Bernstein coefficients, and construct an enclosure for the range of f on X as

$$F_{CTB} = \bar{p}(\mathbf{X}) + R(\mathbf{X})$$

11. RETURN $F_{CTB}(\mathbf{X})$, $\bar{p}(\mathbf{X}), B^{*}, R(\mathbf{X})$, i_f and EXIT.

7 Numerical tests

- The performance of the combined TB form is numerically tested on the same six benchmark examples chosen in ear;ier talk.
- However, the centers of the domains are now different in some examples, while the domains are varied from fairly large to small widths.
- A PC/Pentium III 800 MHz 256 MB RAM machine with a FORTRAN 90 compiler, and version 8.1 of the COSY-INFINITY package of Berz *et al.* are used for the testing.
- In each test example, the following are computed:
  - $F_{TM}(\mathbf{X})$ - using Taylor model of Berz *et al*.
  - $F_{LR}(\mathbf{X})$ - using TB form of Lin and Rokne.
  - $F_{TB}(\mathbf{X})$ - using the improved TB form
  - $F_{CTB}(\mathbf{X})$ - using the combined TB form
  - $F_{inner}(\mathbf{X})$ - using *inner* estimates of the range computed with Moore-Skelboe optimization algorithm.

8 List of Examples

Example 1 Gritton's function in Chemical Engg. $\mathbf{X}^{(i)} = \left[-1 + 2^{-i}\,[-1,1]\right]$

Example 2 Jennrich and Sampson. $\mathbf{X}^{(i)} = \left[-1 + 2^{-i}\,[-1,1]\right]^2$

Example 3 Levy. $\mathbf{X}^{(i)} = \left[9.5 + 2^{-i}\,[-1,1]\right]^3$

Example 4 Trigonometric. $\mathbf{X}^{(i)} = \left[1.75 + 2^{-i}\,[-1,1]\right]^4$

Example 5 Griewank. $\mathbf{X}^{(i)} = \left[0.5 + 2^{-i}\,[-1,1]\right]^5$

Example 6 Trigonometric. $\mathbf{X}^{(i)} = \left[1.75 + 2^{-i}\,[-1,1]\right]^6$

- Information in numerical testing:

$\mathcal{H}$	maximum amount of overestimation
$\mathcal{R}$	ratio of reduction in overestimation
$\mathcal{R}^*$	theoretical value of $\mathcal{R}$
t	computational time
MLL	maximum list length
SD	number of subdivisions

- The quantities MLL and SD are applicable only to the F_{TB} and F_{CTB} forms, where Bernstein subdivision is involved.
- A few sample tables are instead given in this paper, see Tables 1 and 2 .
- The data for F_{LR}, F_{TM} are not given in the table because F_{LR} fails to produce results of this accuracy due to excessive memory demands, whereas F_{TM} requires the domain width parameter $i >> 7$, exceeding the scope of the present investigations.
- In all Tables, the results are suitably rounded for display purposes.

Table 1: Overestimations and their reduction ratios for Taylor order $m = 4$ obtained with F_{TM}, F_{LR}, F_{TB} and F_{CTB} in Example 1 Gritton (1-dimensional).

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$5E+7$
$\mathcal{H}_{LR}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$5E+7$
$\mathcal{H}_{TB}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$5E+7$
$\mathcal{H}_{CTB}$	$2E+34$	$1E+29$	$6E+23$	$5E+18$	$1E+14$	$2E+10$	$5E+7$
$\mathcal{R}^*$	$-$	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	$5E+2$
$\mathcal{R}_{LR}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	$5E+2$
$\mathcal{R}_{TB}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	$5E+2$
$\mathcal{R}_{CTB}$	$-$	$2E+5$	$2E+5$	$1E+5$	$4E+4$	$6E+3$	$5E+2$
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	5	5	5	5	5	5	4
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	43	39	37	35	33	31	29
SD_{CTB}	0	0	0	0	0	0	0

Table 1 (Contd.)

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$8E+5$	$8E+4$	$1E+4$	$4E+3$	$9E+2$	$2E+2$	$2E+1$	$1E+1$
$\mathcal{H}_{LR}$	$7E+5$	$2E+4$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{TB}$	$7E+5$	$2E+4$	$6E+2$	$2E+1$	$5E-1$	$1E-2$	$5E-4$	$1E-5$
$\mathcal{H}_{CTB}$	$7E+5$	$2E+4$	$6E+2$	$2E+1$	$5E-1$	$1E-2$	$5E-4$	$1E-5$
$\mathcal{R}^*$	32	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	57.7	10.8	5.0	4.1	4.0	4.0	4.0	4.0
$\mathcal{R}_{LR}$	71.2	38.0	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	77.2	38.0	35.9	33.9	32.9	32.4	32.2	32.1
$\mathcal{R}_{CTB}$	71.2	38.0	35.9	33.9	32.9	32.4	32.2	32.1
t_{TM}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{LR}	$1E-2$	$1E-2$	$-$	$-$	$-$	$-$	$-$	$-$
t_{TB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
t_{CTB}	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$	$1E-2$
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	3	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	27	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 2: Overestimations and their reduction ratios for Taylor order $m = 4$ obtained with F_{TM}, F_{LR}, F_{TB} and F_{CTB} in Example 6 Trignometric 6 - dimensional.

i	-7	-6	-5	-4	-3	-2	-1
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^7$	$2*2^6$	$2*2^5$	$2*2^4$	$2*2^3$	$2*2^2$	$2*2^1$
$\mathcal{H}_{TM}$	$6E+19$	$6E+16$	$6E+13$	$6E+10$	$1E+8$	$4E+5$	$1E+4$
$\mathcal{H}_{LR}$	$6E+19$	$6E+16$	$6E+13$	$6E+10$	$1E+8$	$4E+5$	$5E+3$
$\mathcal{H}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{CTB}$	$6E+19$	$6E+16$	$6E+13$	$6E+10$	$1E+8$	$4E+5$	$5E+3$
$\mathcal{R}^*$	$-$	32	32	32	32	32	32
$\mathcal{R}_{TM}$	$-$	$1E+3$	983.9	899.3	659.1	251.0	37.1
$\mathcal{R}_{LR}$	$-$	$1E+3$	983.9	899.3	661.1	271.4	66.5
$\mathcal{R}_{TB}$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{CTB}$	$-$	$1E+3$	983.9	899.3	661.1	271.4	66.5
t_{TM}	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{LR}	6.0	6.0	6.0	6.0	6.0	6.0	6.0
t_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
t_{CTB}	6.0	6.0	6.0	6.0	6.0	6.0	6.0
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{CTB}	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{CTB}	0	0	0	0	0	0	0

Table 2 (Contd.)

i	0	1	2	3	4	5	6	7
$w\left(\mathbf{X}^{(i)}\right)$	$2*2^{-0}$	$2*2^{-1}$	$2*2^{-2}$	$2*2^{-3}$	$2*2^{-4}$	$2*2^{-5}$	$2*2^{-6}$	$2*2^{-7}$
$\mathcal{H}_{TM}$	$1E+3$	$3E+2$	$6E+1$	$1E+1$	$3E+0$	$1E+0$	$2E-1$	$5E-2$
$\mathcal{H}_{LR}$	$4E+1$	$7E-1$	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{H}_{TB}$	$4E+1$	$7E-1$	$1E-2$	$2E-4$	$5E-6$	$2E-7$	$3E-9$	$1E-10$
$\mathcal{H}_{CTB}$	$4E+1$	$7E-1$	$1E-2$	$2E-4$	$5E-6$	$2E-7$	$3E-9$	$1E-10$
$\mathcal{R}^*$	32	32	32	32	32	32	32	32
$\mathcal{R}_{TM}$	7.1	5.0	4.5	4.3	4.1	4.1	4.0	4.0
$\mathcal{R}_{LR}$	128.2	61.3	$-$	$-$	$-$	$-$	$-$	$-$
$\mathcal{R}_{TB}$	$-$	61.3	56.9	51.6	46.2	41.2	37.1	28.5
$\mathcal{R}_{CTB}$	128.2	61.3	56.9	51.6	46.2	41.2	37.1	28.5
t_{TM}	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$	$4E-2$
t_{LR}	6.0	6.0	$-$	$-$	$-$	$-$	$-$	$-$
t_{TB}	7.1	7.1	7.1	7.1	7.1	7.1	7.1	7.1
t_{CTB}	6.0	6.0	7.0	7.0	7.0	7.0	7.0	7.0
MLL_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
MLL_{TB}	0	0	0	0	0	0	0	0
MLL_{CTB}	0	0	0	0	0	0	0	0
SD_{TM}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{LR}	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
SD_{TB}	0	0	0	0	0	0	0	0
SD_{CTB}	0	0	0	0	0	0	0	0

Table 3: Domain width parameter i and time taken by Algorithms TB and CTB to reach an accuracy of $1E-10$ for various Taylor orders in Examples 1 to 6.

Example	Name	Order, m	F_{TB}		F_{CTB}	
			i	time	i	time
1	Gritton	2	>7	$-$	>7	$-$
		4	>7	$-$	>7	$-$
		6	7	0.15	7	0.15
		8	5	0.13	5	0.13
2	Jennrich & Sampson	2	>7	$-$	>7	$-$
		4	7	0.27	7	0.23
		6	5	0.44	5	0.33
		8	4	0.65	4	0.42
3	Levy	2	>7	$-$	>7	$-$
		4	7	2.17	7	0.15
		6	5	4.45	5	0.13
		8	3	4.09	3	0.35
4	Trigonometric	2	>7	$-$	>7	$-$
		4	7	153	7	0.39
		6	5	1712	5	3.0
		8	3	4.87 hr	3	15.6
5	Griewank	2	>7	∞	>7	$-$
		4	6	∞	6	7.2
		6	4	∞	4	91.5
		8	3	∞	3	909
6	Trigonometric	2	>7	∞	>7	$-$
		4	7	∞	7	96
		6	5	∞	5	3679
		8	3	∞	3	16.5 hr

9 Observations

From the results given in the Tables, we observe that

- With Taylor model, only *quadratic* convergence is obtained in all problems, irrespective of the chosen Taylor order m.
- For large domains, F_{TB} requires comparatively large times and memory than F_{LR}, and fails to compute the range enclosure for higher dimensional problems; F_{LR} does not fail for any problem dimension.
- For small domains, the situation is the reverse: F_{LR} fails for any problem dimension, while F_{TB} does not fail for any problem dimension.
- In view of the above, the practical utility of F_{LR} is severely restricted as an inclusion form for obtaining higher order convergence.
- Further, F_{TB} is not satisfactory for large domain widths.
- Hence, neither of the existing TB forms is really satisfactory in applications where the domain is made to shrink from large to small widths using techniques such as subdivision.

10 Observations (Contd.)

- The proposed form F_{CTB} does not fail to compute the range enclosure for any domain width and for any problem dimension !
- With F_{CTB} as an inclusion function form, higher order convergence of orders up to 9 is quite easily obtained in problems of up to $4-\mathrm{dim}$.
- In problems of 5 and $6-\mathrm{dim}$, higher order convergence of orders up to 9 is again obtained;
- But, computational demands are somewhat large for the $5-\mathrm{dim}$ problem, and become excessive for the $6-\mathrm{dim}$ one.
- This behavior is identical to that for F_{TB} - it is expected to be so, because F_{CTB} reduces to F_{TB} for small domains.
- With F_{TB}, the maximum list length for large domains increases with the problem dimension, and becomes excessive for the $6-\mathrm{dim}$ example. As the domain width decreases, the maximum list length with F_{TB} decreases and tends to zero.
- On the other hand, the maximum list length with the proposed form F_{CTB} is *nil*, for any domain width and for any problem dimension !

- The computation time taken for F_{CTB} is smaller than that for F_{TB} by as much as $12,000$ times for large domain widths, by as much as 73 times for intermediate domain widths, and is the same as for F_{TB} for small domain widths.
- The overestimation given by F_{CTB} is usually about the same as that given by F_{TB} - of course, in those cases where F_{TB} succeeds at all in computing the range enclosures. A similar remark holds with respect to the overestimation given by F_{LR}.
- The best performance in terms of both the computational time and domain width parameter i is given by the proposed form F_{CTB}, for Taylor order $m = 4$.

11 Conclusions

- In all examples considered, the new combined TB form indeed numerically exhibited the higher order convergence property.
- The overestimation given by the proposed TB form was usually about the same as that given by existing TB forms - in all those cases where the latter forms succeed in computing the range enclosures.
- Whereas the existing TB forms failed to compute the range enclosures for some domain widths, the proposed TB form did not fail to compute the same for any domain width and for any problem dimension.
- The maximum list length needed by the proposed TB form was *nil*, for any domain width and for any problem dimension.
- For large to intermediate domain widths, the proposed TB form was significantly faster than F_{TB}, by as much as $2-4$ orders of magnitude.