The bifurcation graph of the Kuramoto-Sivashinski equation

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The Kuramoto-Sivashinsky equation

\[ v_t + 4\Delta^2 v + \alpha \left( \Delta v + \frac{1}{2} (\nabla v)^2 \right) = 0, \quad \alpha > 0 \]

The unidimensional equation

\[ v_t + 4v_{xxxx} + \alpha \left( v_{xx} + \frac{1}{2}v_x^2 \right) = 0 \quad 0 \leq x \leq 2\pi \]

has been the object of many analytical and numerical studies. Differentiating the equation with respect to \( x \) and setting \( u = v_x \) one obtains the alternative form of the equation

\[ u_t + 4u_{xxxx} + \alpha(u_{xx} + (u^2)_x) = 0 \quad 0 \leq x \leq \pi \, , \, u(0, t) = u(\pi, t) = 0 \, , \, \forall t \]

We focus our attention on the bifurcation graph of odd steady states, i.e. solutions of the problem

\[ 4u^{(4)} + \alpha(u'' + 2uu') = 0 \quad 0 \leq x \leq \pi \, , \, u(0) = u(\pi) = 0 . \]
In P. Zgliczyński, K. Mischaikow, *Rigorous numerics for partial differential equations: The Kuramoto-Sivashinsky equation*, Found. of Comp. Math. 1 255-288 (2001) a new method for proving existence theorems for nonlinear dissipative equations has been introduced. This method relies on the existence of self consistent a priori bounds.

In P. Zgliczyński, K. Mischaikow, *Towards a rigorous steady states bifurcation diagram for the Kuramoto-Sivashinski equation - a computer assisted rigorous approach* the problem of the rigorous study of a bifurcation diagram has been set.
Main features of the equation.

Let $R_1 : u(x) \mapsto -u(\pi - x)$. It is clear that if $u$ is a solution, then $R_1(u)$ is also a solution. It is also clear that, if $u$ is a symmetric solution with respect to $R_1$, i.e. $u = R_1(u)$, then

$$-u(\pi/2 - x) \text{ if } 0 \leq x \leq \pi/2 - u(3\pi/2 - x) \text{ if } \pi/2 \leq x \leq \pi$$

is also a solution. In a similar fashion one can define the symmetry $R_{2k}$ for all $k = 0, 1, \ldots$ and verify that if $u$ is a solution satisfying $u = R_{2k}(u)$ for all $k = 0, \ldots, n$ and $u \neq R_{2n+1}(u)$ then $R_{2n+1}(u)$ is another solution. This means that all solutions come in pairs.
Consider the linearization of the equation at 0

\[ L_0(u) = -4u_{xxxx} - \alpha u_{xx} = 0, \quad 0 \leq x \leq \pi \quad u(0) = u(\pi) = 0. \]

The eigenvalues are \( \lambda_k = \alpha k^2 - 4k^4, k = 1, 2, \ldots \). If \( \alpha < 4 \), then all eigenvalues are negative and no nontrivial steady state exists. At \( \alpha = 4k^2 \), \( k = 1, 2, \ldots \), \( L_0 \) is not invertible and the full equation admits a pitchfork bifurcation at 0 and two branches of nontrivial solutions bifurcate from the zero solution. These branches are called unimodal \( (k = 1) \), bimodal \( (k = 2) \), and generically \( k \)-modal.

If \( \alpha \neq 4k^2, k = 1, 2, \ldots \), then \( L_0 \) is invertible and therefore there cannot be nontrivial solutions branching off the trivial solution other than the \( k \)-modal branches.
Given $\rho > 0$, let $\mathcal{D}_\rho = \{ x \in \mathbb{C} : |\text{Im}(x)| < \rho \}$, and denote by $\mathcal{C}_\rho$ the space of all functions $f : \mathcal{D}_\rho \rightarrow \mathbb{C}$,

$$f(x) = \sum_{k=1}^{\infty} f_n \sin(kx) + \sum_{k=0}^{\infty} f_n' \cos(kx), \quad x \in \mathcal{D}_\rho,$$

which take real values when restricted to $\mathbb{R}$, and for which the norm

$$\|f\|_\rho = \sum_{k=1}^{\infty} |f_k| \rho^k + \sum_{k=0}^{\infty} |f_n'| \rho^k$$

is finite. The subspace of odd and even functions in $\mathcal{C}_\rho$ will be denoted by $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$, respectively.

Our first main result concerns the existence of branches of solutions and of bifurcation points.
Theorem 1. The stationary Kuramoto-Sivashinski equation admits 10 pitchfork bifurcation points \((4, 16, 36, 64, A, B, E, F, G, H)\), 4 intersection bifurcation points \((C_\pm, D_\pm)\), 6 fold bifurcation points \((34_\pm, 36_\pm, 49_\pm)\). Such bifurcations are linked by continuous branches of solutions. There exist no other branches bifurcating from the trivial branch when \(\alpha \in [0, 80]\) and no other bifurcations. All such solutions are in \(A_\rho\), with \(\rho = 1/16\). Furthermore, the value of the parameter \(\alpha\) where the bifurcations occur is given in the following table.
<table>
<thead>
<tr>
<th>bifurcation</th>
<th>$\alpha$</th>
<th>type</th>
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<tbody>
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<td>4</td>
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<tr>
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</tr>
<tr>
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</tr>
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<td>$D_{\pm}$</td>
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</tr>
<tr>
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<td>reverse fold</td>
</tr>
<tr>
<td>$49_{\pm}$</td>
<td>49.66453...</td>
<td>fold</td>
</tr>
</tbody>
</table>
The second results concerns the stability of stationary solutions under the flow of the full Kuramoto-Sivashinski equation.

Let \( t \mapsto v \in \mathcal{A}_\rho \) be a stationary solution. Then the time evolution of a function \( u = v + h \) is described by the equation

\[
\dot{h} = L_v h + \alpha D(h^2), \quad L_v h = -(4D^4 + \alpha D^2) h + 2\alpha D(vh).
\]

Let \( \ell \) be the number of eigenvalues of \( L_v \) that have a positive real part, and assume that \( L_v \) has no eigenvalues on the imaginary axis. Then the map \( \Phi^1 \) on \( \mathcal{A}_\rho \) has smooth local stable and unstable manifolds at \( v \), with the unstable manifold being of dimension \( \ell \) and tangent at \( v \) to the spectral subspace of \( L_v \) corresponding to the \( \ell \) eigenvalues with positive real part.
Our goal is to determine $\ell$ for some stationary solutions, e.g. for all solutions belonging to the branches obtained in with $\alpha \in \{10, 20, \ldots, 80\}$.

**Theorem 2.** For all pairs $(\alpha, B)$ listed in the next table, the Kuramoto-Sivashinski equation admits a stationary solution on the branch $B$ and the unstable manifold at this solution has dimension $\ell$. 
<table>
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<tr>
<th>$\alpha$</th>
<th>B</th>
<th>$\ell$</th>
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<tr>
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</tr>
<tr>
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</tr>
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</tr>
<tr>
<td>40</td>
<td>3−</td>
<td>1</td>
</tr>
</tbody>
</table>
Main ideas of the proofs

We rewrite the equation in the form $F_\alpha(u) = u$,

$$F_\alpha(u) = -\frac{\alpha}{4} D^{-2} u + \frac{\alpha}{4} D^{-3} (u^2),$$

where $D^{-1}$ denotes the antiderivative operator on the space of continuous $2\pi$-periodic functions with average zero, extended to functions with nonzero average by first subtracting their average value.

We focus here on cases where the spectrum of $DF_\alpha(u)$ is bounded away from 1.
We choose a finite dimensional approximation $M$ for the map 
$[DF_\alpha(u_0) - I]^{-1} - I$, and then define

$$C_\alpha(u) = F_\alpha(u) - M[F_\alpha(u) - u].$$

Formally, the map $C$ is close to the Newton map for $F_\alpha$. Thus, our goal is to prove that

$$\|C_\alpha(u_0) - u_0\| < \varepsilon, \quad \|DC_\alpha(u)\| < K, \quad \varepsilon + Kr < r,$$

for some real numbers $r, \varepsilon, K > 0$, and for arbitrary $u$ in a closed ball $B$ of radius $r$, centered at $u_0$. Then the contraction principle implies that $F$ has a unique fixed point $U$ in $B$, and if $M - I$ is invertible, then $u$ is also a fixed point of $F$ and thus a solution.
Proposition 1. Let $N \in \mathbb{Z}$, let $\{(u_i, \varepsilon_i)\} \in A \times \mathbb{R}$, $i = 0, \ldots, 2N$ and 
$\{\alpha_i \in \mathbb{R}\}, i = 0, \ldots, N$. Assume that for all $i = 0, \ldots, N$ there exists a unique
solution with $\alpha = \alpha_i$ in $B(u_{2i}, \varepsilon_{2i})$ and that for all $i = 0, \ldots, N - 1$ and all
$\alpha \in [\alpha_i, \alpha_{i+1}]$ there exists a unique solution in $B(u_{2i+1}, \varepsilon_{2i+1})$. Assume also
that for all $i = 0, \ldots, N - 1$
$B(u_{2i}, \varepsilon_{2i}) \cup B(u_{2i+2}, \varepsilon_{2i+2}) \subset B(u_{2i+1}, \varepsilon_{2i+1})$. Then there exists a
continuous branch of solutions linking $u_0$ with $u_{2N}$. 
For the study of bifurcations, we write the equation as $\mathcal{F}(\alpha, u) = 0$, where

$$\mathcal{F}(\alpha, u) = -u - \frac{\alpha}{4} D^{-2} u + \frac{\alpha}{4} D^{-3} (u^2).$$

The types of bifurcations considered here take place in two dimensional submanifolds of $\mathcal{A}_\rho$. We will parametrize these surfaces by using the frequency $\alpha$ and the value $\lambda$ of some coordinate function on $\mathcal{A}_\rho$. 
As a coordinate function, we choose a suitable one-dimensional projection \( \ell \propto 0 \). Then we define a two-parameter family of functions \( u(\alpha, \lambda) \) in \( \mathcal{A}_\rho \) by solving

\[
(I - \ell)F(\alpha, u(\alpha, \lambda)) = 0, \quad \ell u(\alpha, \lambda) = \lambda \hat{u},
\]

where \( \hat{u} \) is a fixed nonzero function in the range of \( \ell \). Our goal is to show that for certain rectangles \( I \times J \) in parameter space, the equation has a smooth and locally unique solution \( u : I \times J \to \mathcal{A}_\rho \). Then locally, the solutions of \( F_\alpha(u) = 0 \) are determined by the zeros of the function \( g \),

\[
g(\alpha, \lambda)\hat{u} = \ell F(\alpha, u(\alpha, \lambda)).
\]
The equation for $u = u(\alpha, \lambda)$ is equivalent to the fixed point equation for the map $F_{\alpha, \lambda}$, defined by

$$F_{\alpha, \lambda}(u) = (I - \ell)F_{\alpha}(u) + \lambda \hat{u}.$$ 

This fixed point problem is solved by converting it to a fixed point problem for a map $C_{\alpha, \lambda}$, which is obtained from $F_{\alpha, \lambda}$ in the same way that $C_{\alpha}$ was obtained from $F_{\alpha}$. These estimates imply also that $DF_{\alpha, \lambda}(u)$ is invertible, for all $u$ in the ball being considered. Thus, since $F_{\alpha, \lambda}(u)$ is a polynomial in $\alpha$, $\lambda$, and $u$, the implicit function theorem guarantees that the solution $u = u(\alpha, \lambda)$ depends smoothly on the parameters $\alpha$ and $\lambda$. 
This leaves the problem of verifying that a certain type of bifurcation occurs. For the sake of definiteness, we will restrict our discussion here to the case of a pitchfork bifurcation. A sufficient set of conditions for the existence of such a bifurcation is given below. A concrete example of a function $g$ that satisfies these conditions (near the origin) is $(\alpha, \lambda) \mapsto \lambda^3 - \alpha \lambda$.

If $f$ is any differentiable function of two variables, denote by $\hat{f}$ and $f'$ the partial derivatives of $f$ with respect to the first and second argument, respectively.
Let $I = [\alpha_1, \alpha_2]$ and $J = [-b, b]$.

**Proposition 2.** Let $g$ be a real-valued $C^3$ function on an open neighborhood of $I \times J$, such that $g(\alpha, 0) = 0$ for all $\alpha \in I$, and

1. $g''' > 0$ on $I \times J$,
2. $g' < 0$ on $I \times J$,
3. $g'(\alpha_1, 0) \pm \frac{1}{2} bg''(\alpha_1, 0) > 0$,
4. $\pm g(\alpha_2, \pm b) > 0$,
5. $g'(\alpha_2, 0) < 0$.

Then $g(\alpha, \lambda) = \lambda G'(\alpha, \lambda)$, and the solution set of $G'(\alpha, \lambda) = 0$ in $I \times J$ is the graph of a $C^2$ function $\alpha = a(\lambda)$, defined on a proper subinterval $J_0$ of $J$. This function takes the value $\alpha_2$ at the endpoints of $J_0$, and satisfies $\alpha_1 < a(\lambda) < \alpha_2$ at all interior points of $J_0$, which includes the origin.
The role of the computer

The proofs are based on a discretization of the problem, carried out and controlled with the aid of a computer.

At the trivial level of real numbers, the discretization is implemented by using interval arithmetic. In particular, a number $s \in \mathbb{R}$ is “represented” by an interval $S = [S^-, S^+]$ containing $s$, whose endpoints belong to some finite set of real numbers that are representable on the computer. Such an interval will be called a “standard set” for $\mathbb{R}$. 
The collection of all standard sets for $\mathbb{R}$ will be denoted by $\text{std}(\mathbb{R})$. In what follows, a “bound” on a function $g : X \rightarrow Y$ is a map $G$, from a subset $D_G$ of $\text{std}(X)$ to $\text{std}(Y)$, with the property that $g(s)$ belongs to $G(S)$ whenever $s \in S \in D_G$.

Bounds on the basic arithmetic operation like $(r, s) \rightarrow rs$ are easy to implement on modern computers.

The goal now is to combine these elementary bounds to obtain e.g. a bound $G_1$ on the norm function on $\mathcal{A}_\rho$, and a bound $G_2$ on the map $\mathcal{C}$. Then, in order to prove the first inequality it suffices to verify that $G_1(G_2(S)) \subset U$, where $S$ is a set in $\text{std}(\mathcal{A}_{\rho}^P)$ containing $g_0$, and $U$ is an interval in $\text{std}(\mathbb{R})$ with $U^+ < \varepsilon$. 
To prove the stability result, we write $L_{\nu}$ as a perturbation of a linear operator whose eigenvalues and eigenvectors are known explicitly. Let $T$ and $L = T + A$ be closed linear operators whose spectrum consists of isolated eigenvalues only. Then the following holds.

**Proposition 3.** Let $\Omega$ be a bounded open subset of $\mathbb{C}$, whose boundary $\partial \Omega$ consists of finitely many rectifiable Jordan curves and avoids the eigenvalues of $T$. If

$$\|A(T - z)^{-1}\| < 1, \quad \forall z \in \partial \Omega,$$

then $T$ and $L = T + A$ have the same number of eigenvalues (counting multiplicities) in the region $\Omega$, and in its closure.

A proof of this (well known) fact is based on the integral formula

$$P_{\Omega} = \frac{1}{2\pi i} \int_{\Gamma} (\mathcal{L} - z)^{-1} dz = \frac{1}{2\pi i} \int_{\Gamma} (T - z)^{-1} [I + A(T - z)^{-1}]^{-1} dz$$

for the spectral projection $P_{\Omega}$ of $\mathcal{L}$, associated with the eigenvalues of $\mathcal{L}$ in $\Omega$. 
This proposition will be applied not to $L_v$ directly, but to the operator

$$\mathcal{L} = E^{-1}L_vE,$$

where $E \simeq I$ is a suitable linear isomorphism of $A_{\rho}$. Here, and in what follows, $U \simeq V$ means that $U - V$ is finite-dimensional.

**Corollary 1.** Let $T \simeq L_0$ and $S \simeq L_0$ be linear operators on $A_{\rho}$, that have no eigenvalues on the imaginary axis. Let $A = \mathcal{L} - T$. If

$$\|AS^{-1}\| < 1, \quad \|S(T - iy)^{-1}\| \leq 1, \quad \forall y \in \mathbb{R},$$

then $L_v$ and $T$ have the same number of eigenvalues in the halfplane $\text{Im}(z) > 0$, and in its closure.
In our proof the isomorphism $E$ is chosen in such a way that $\mathcal{L}$ is close to an operator $\mathcal{T}$ that is block-diagonal, in the sense that all eigenvectors of $\mathcal{T}$ are either Fourier monomials (for real eigenvalues) or a linear combination of two Fourier monomials (for pairs of complex conjugate eigenvalues). In order to simplify our estimates, the operator $\mathcal{S}$ is taken to be diagonal. Given $\mathcal{T}$, it is easy to find such an operator $\mathcal{S}$ that satisfies $\|\mathcal{S}(\mathcal{T} - iy)^{-1}\| \leq 1$ without being smaller than necessary.