The bifurcation graph of the Kuramoto-Sivashinski equation

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The Kuramoto-Sivashinsky equation

$$v_t + 4\Delta^2 v + \alpha\left(\Delta v + \frac{1}{2}(\nabla v)^2\right) = 0, \qquad \alpha > 0$$

was introduced by Kuramoto and Tsuzuki in *Persistent propagation of concentration waves in dissipative media far from thermal equilibrium*, Progr. Theor. Phys, 55 365-369 (1976) in the context of turbulence for a system of reaction diffusion equations and by Sivashinsky in *Nonlinear analysis of hydrodynamic instability in laminal flames - I. Derivation of basic equations*, Acta Astr. 4 1177-1206 (1977) to model thermal diffusive instabilities in laminar flame fronts.

The unidimensional equation

$$v_t + 4v_{xxxx} + \alpha\left(v_{xx} + \frac{1}{2}v_x^2\right) = 0 \qquad 0 \leq x \leq 2\pi$$

has been the object of many analytical and numerical studies. Differentiating the equation with respect to x and setting $u = v_x$ one obtains the alternative form of the equation

$$u_t + 4u_{xxxx} + \alpha(u_{xx} + (u^2)_x) = 0 \qquad 0 \leq x \leq \pi\,, u(0,t) = u(\pi,t) = 0\,, \forall t$$

We focus our attention on the bifurcation graph of odd steady states, i.e. solutions of the problem

$$4u^{(4)} + \alpha(u'' + 2uu') = 0 \quad 0 \leq x \leq \pi\,, \quad u(0) = u(\pi) = 0\,.$$

In P. Zgliczyński, K. Mischaikow, *Rigorous numerics for partial differential equations: The Kuramoto-Sivashinsky equation*, Found. of Comp. Math. 1 255-288 (2001) a new method for proving existence theorems for nonlinear dissipative equations has been introduced. This method relies on the existence of self consistent a priori bounds.

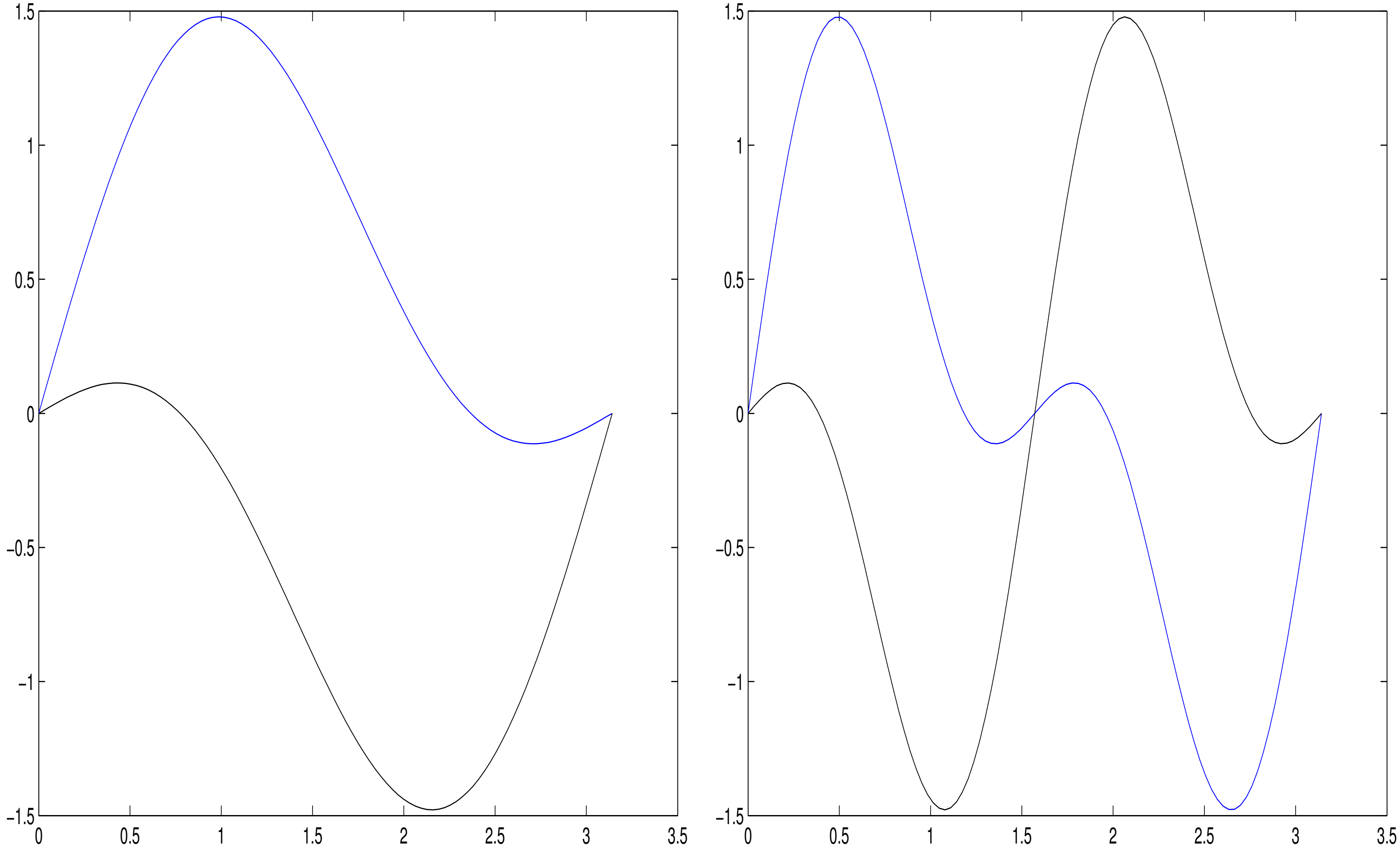
In P. Zgliczyński, K. Mischaikow, *Towards a rigorous steady states bifurcation diagram for the Kuramoto-Sivashinski equation - a computer assisted rigorous approach* the problem of the rigorous study of a bifurcation diagram has been set.

Main features of the equation.

Let $R_1 : u(x) \mapsto -u(\pi - x)$. It is clear that if u is a solution, then $R_1(u)$ is also a solution. It is also clear that, if u is a symmetric solution with respect to R_1, i.e. $u = R_1(u)$, then

$$-u(\pi/2 - x) \text{ if } 0 \leq x \leq \pi/2 - u(3\pi/2 - x) \text{ if } \pi/2 \leq x \leq \pi$$

is also a solution. In a similar fashion one can define the symmetry R_{2^k} for all $k = 0, 1, \ldots$ and verify that if u is a solution satisfying $u = R_{2^k}(u)$ for all $k = 0, \ldots, n$ and $u \neq R_{2^{n+1}}(u)$ then $R_{2^{n+1}}(u)$ is another solution. This means that all solutions come in pairs.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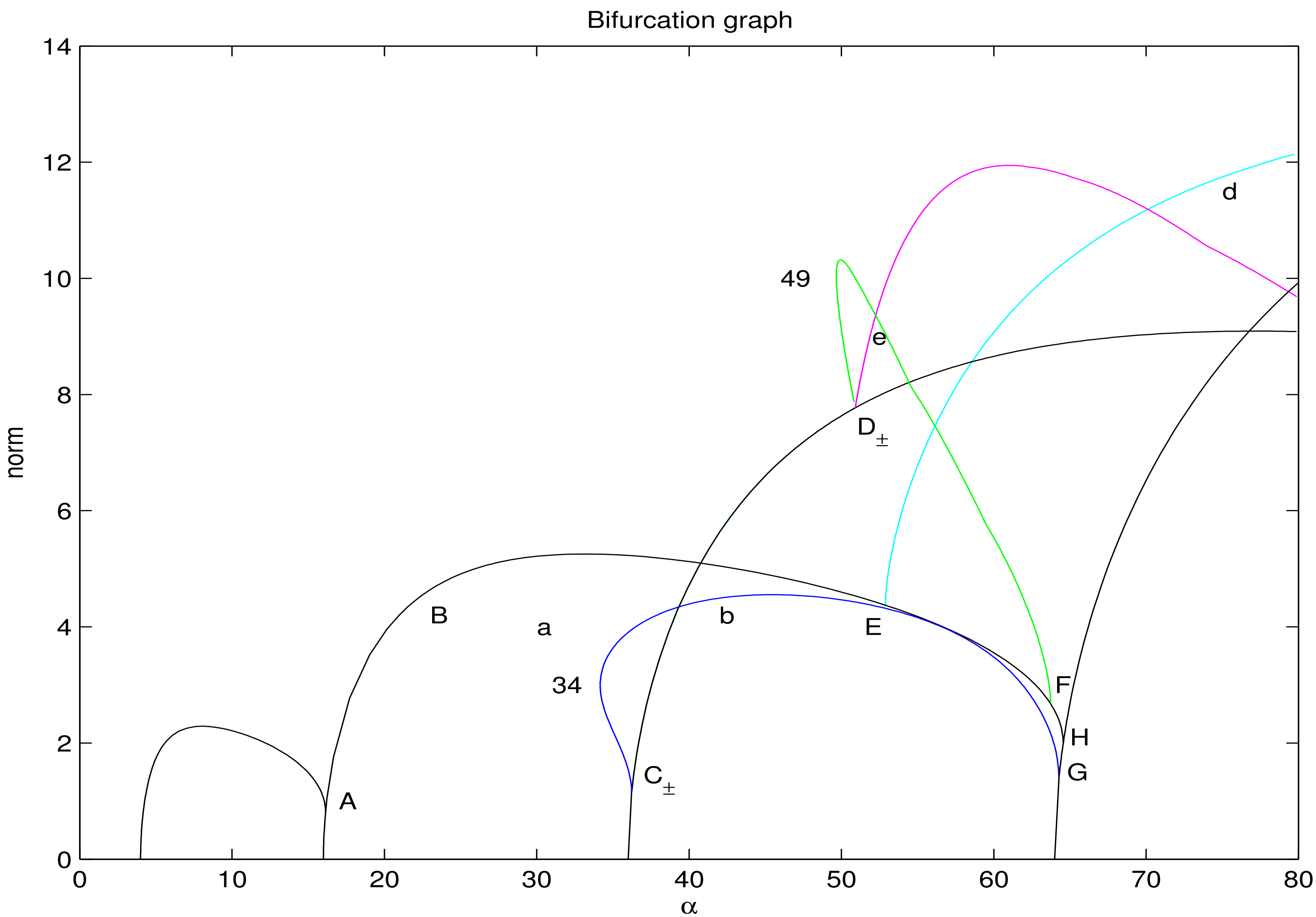
Consider the linearization of the equation at 0

$$L_0(u) = -4u_{xxxx} - \alpha u_{xx} = 0, \qquad 0 \le x \le \pi \qquad u(0) = u(\pi) = 0\,.$$

The eigenvalues are $\lambda_k = \alpha k^2 - 4k^4$, $k = 1, 2, \ldots$. If $\alpha < 4$, then all eigenvalues are negative and no nontrivial steady state exists. At $\alpha = 4k^2$, $k = 1, 2, \ldots$, L_0 is not invertible and the full equation admits a pitchfork bifurcation at 0 and two branches of nontrivial solutions bifurcate from the zero solution. These branches are called unimodal ($k = 1$), bimodal ($k = 2$), and generically $k-$modal.

If $\alpha \neq 4k^2$, $k = 1, 2, \ldots$, then L_0 is invertible and therefore there cannot be nontrivial solutions branching off the trivial solution other than the $k-$modal branches.


Bifurcation graph
norm
α
0
2
4
6
8
10
12
14
0
10
20
30
40
50
60
70
80
A
B
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34
C±
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E
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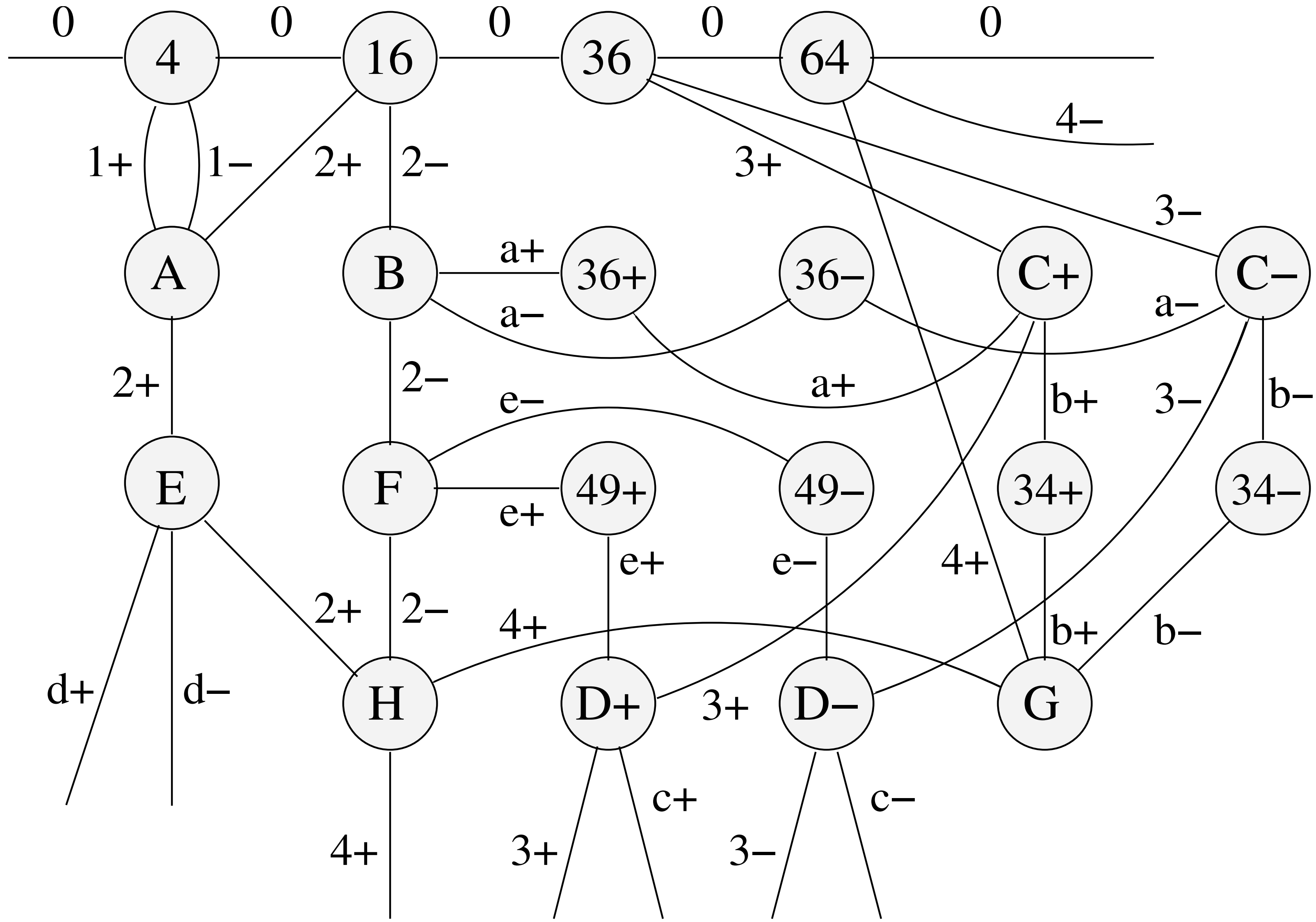
G
D±
e
49
d


0
4
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16
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36
0
64
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4−
1+
1−
2+
2−
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3−
A
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36+
36−
C+
C−
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2+
2−
e−
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E
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e+
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2+
2−
4+
b+
b−
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Given $\rho > 0$, let $\mathcal{D}_\rho = \{x \in \mathbb{C} \colon |\mathrm{Im}(x)| < \rho\}$, and denote by $\mathcal{C}_\rho$ the space of all functions $f : \mathcal{D}_\rho \to \mathbb{C}$,

$$f(x) = \sum_{k=1}^{\infty} f_n \sin(kx) + \sum_{k=0}^{\infty} f'_n \cos(kx) \,, \qquad x \in \mathcal{D}_\rho \,,$$

which take real values when restricted to $\mathbb{R}$, and for which the norm

$$\|f\|_\rho = \sum_{k=1}^{\infty} |f_k| \rho^k + \sum_{k=0}^{\infty} |f'_n| \rho^k$$

is finite. The subspace of odd and even functions in $\mathcal{C}_\rho$ will be denoted by $\mathcal{A}_\rho$ and $\mathcal{B}_\rho$, respectively.

Our first main result concerns the existence of branches of solutions and of bifurcation points.

Theorem 1. *The stationary Kuramoto-Sivashinski equation admits 10 pitchfork bifurcation points ($4, 16, 36, 64, A, B, E, F, G, H$), 4 intersection bifurcation points ($C_\pm$ and $D_\pm$), 6 fold bifurcation points ($34_\pm, 36_\pm$ and $49_\pm$). Such bifurcations are linked by continuous branches of solutions. There exist no other branches bifurcating from the trivial branch when $\alpha \in [0, 80]$ and no other bifurcations. All such solutions are in $\mathcal{A}_\rho$, with $\rho = 1/16$. Furthermore, the value of the parameter α where the bifurcations occur is given in the following table.*

bifurcation	α	type
4	4	pitchfork
16	16	pitchfork
36	36	pitchfork
64	64	pitchfork
A	$16.13985\ldots$	reverse pitchfork
B	$22.55606\ldots$	reverse pitchfork
$C_\pm$	$36.23390\ldots$	intersection
$D_\pm$	$50.90983\ldots$	intersection
E	$52.89105\ldots$	pitchfork
F	$63.73699\ldots$	reverse pitchfork
G	$64.27481\ldots$	reverse pitchfork
H	$64.55942\ldots$	reverse pitchfork
$34_\pm$	$34.16913\ldots$	fold
$36_\pm$	$36.23501\ldots$	reverse fold
$49_\pm$	$49.66453\ldots$	fold

The second results concerns the stability of stationary solutions under the flow of the full Kuramoto-Sivashinski equation.

Let $t \mapsto v \in \mathcal{A}_\rho$ be a stationary solution. Then the time evolution of a function $u = v + h$ is described by the equation

$$\dot{h} = L_v h + \alpha D(h^2)\,, \qquad L_v h = -\bigl(4D^4 + \alpha D^2\bigr)h + 2\alpha D(vh)\,.$$

Let ℓ be the number of eigenvalues of L_v that have a positive real part, and assume that L_v has no eigenvalues on the imaginary axis. Then the map Φ^1 on $\mathcal{A}_\rho$ has smooth local stable and unstable manifolds at v, with the unstable manifold being of dimension ℓ and tangent at v to the spectral subspace of L_v corresponding to the ℓ eigenvalues with positive real part.

Our goal is to determine ℓ for some stationary solutions, e.g. for all solutions belonging to the branches obtained in with $\alpha \in \{10, 20, \ldots, 80\}$.

Theorem 2. *For all pairs (α, B) listed in the next table, the Kuramoto-Sivashinski equation admits a stationary solution on the branch B and the unstable manifold at this solution has dimension ℓ.*

α	B	ℓ
10	1+	0
10	1−	0
20	2+	0
20	2−	1
30	2+	0
30	2−	0
30	$a+$	1
30	$a-$	1
40	2+	0
40	2−	2
40	3+	1
40	3−	1

Main ideas of the proofs

We rewrite the equation in the form $F_\alpha(u) = u$,

$$F_\alpha(u) = -\frac{\alpha}{4} D^{-2} u + \frac{\alpha}{4} D^{-3}\left(u^2\right),$$

where D^{-1} denotes the antiderivative operator on the space of continuous 2π-periodic functions with average zero, extended to functions with nonzero average by first subtracting their average value.

We focus here on cases where the spectrum of $DF_\alpha(u)$ is bounded away from 1.

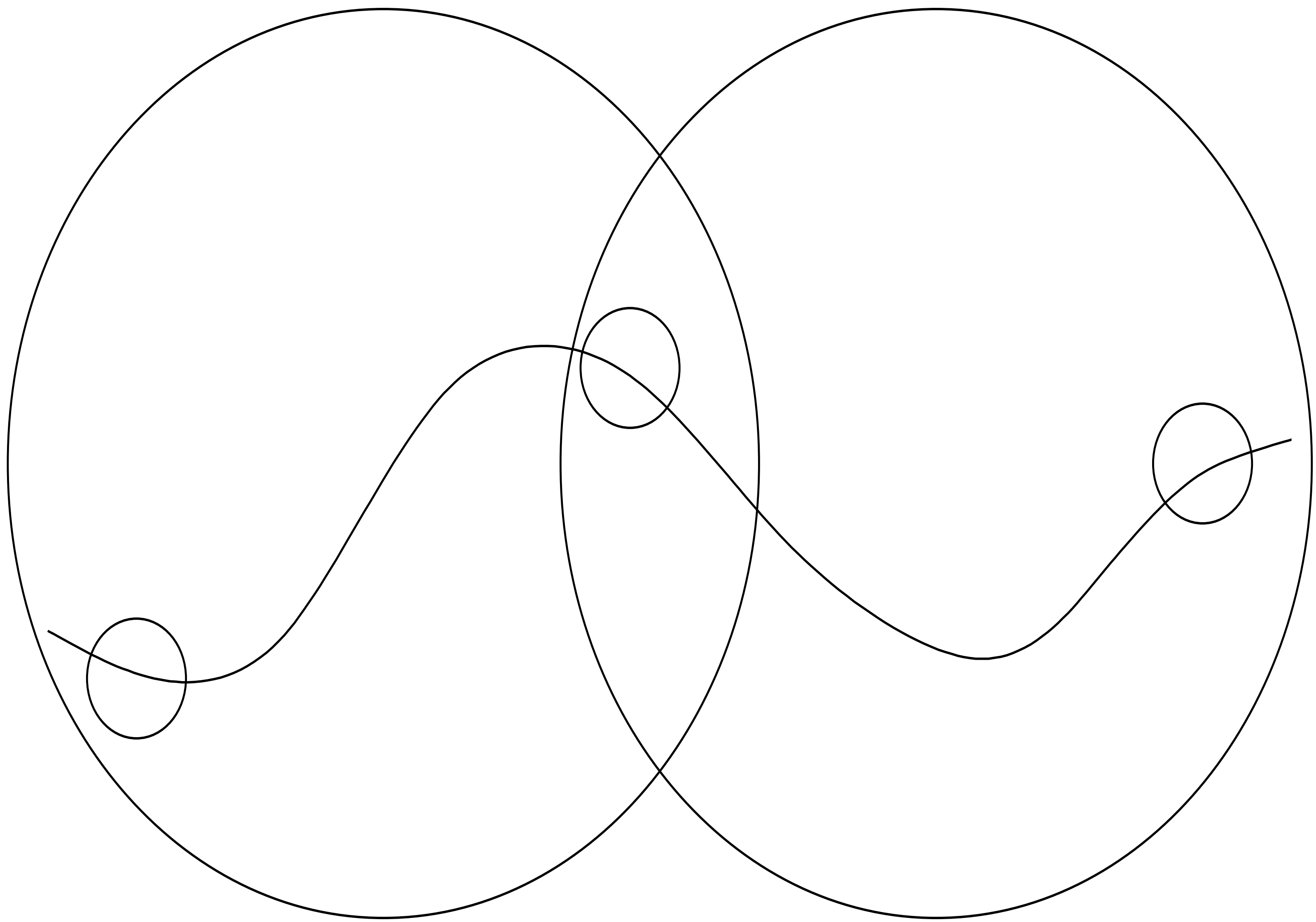
We choose a finite dimensional approximation M for the map $[DF_\alpha(u_0) - \mathrm{I}]^{-1} - \mathrm{I}$, and then define

$$\mathcal{C}_\alpha(u) = F_\alpha(u) - M\big[F_\alpha(u) - u\big].$$

Formally, the map $\mathcal{C}$ is close to the Newton map for F_α . Thus, our goal is to prove that

$$\|\mathcal{C}_\alpha(u_0) - u_0\| < \varepsilon\,, \qquad \|D\mathcal{C}_\alpha(u)\| < K\,, \qquad \varepsilon + Kr < r\,,$$

for some real numbers $r, \varepsilon, K > 0$, and for arbitrary u in a closed ball B of radius r, centered at u_0 . Then the contraction principle implies that F has a unique fixed point U in B, and if $M - \mathrm{I}$ is invertible, then u is also a fixed point of F and thus a solution.



Proposition 1. *Let* $N \in \mathbb{Z}$*, let* $\{(u_i, \varepsilon_i)\} \in \mathcal{A} \times \mathbb{R}$*,* $i = 0, \ldots, 2N$ *and* $\{\alpha_i \in \mathbb{R}\}$*,* $i = 0, \ldots, N$*. Assume that for all* $i = 0, \ldots, N$ *there exists a unique solution with* $\alpha = \alpha_i$ *in* $B(u_{2i}, \varepsilon_{2i})$ *and that for all* $i = 0, \ldots, N-1$ *and all* $\alpha \in [\alpha_i, \alpha_{i+1}]$ *there exists a unique solution in* $B(u_{2i+1}, \varepsilon_{2i+1})$*. Assume also that for all* $i = 0, \ldots, N-1$

$B(u_{2i}, \varepsilon_{2i}) \cup B(u_{2i+2}, \varepsilon_{2i+2}) \subset B(u_{2i+1}, \varepsilon_{2i+1})$*. Then there exists a continuous branch of solutions linking* u_0 *with* u_{2N}*.*

For the study of bifurcations, we write the equation as $\mathcal{F}(\alpha, u) = 0$, where

$$\mathcal{F}(\alpha, u) = -u - \tfrac{\alpha}{4} D^{-2} u + \tfrac{\alpha}{4} D^{-3}(u^2)\,.$$

The types of bifurcations considered here take place in two dimensional submanifolds of $\mathcal{A}_\rho$. We will parametrize these surfaces by using the frequency α and the value λ of some coordinate function on $\mathcal{A}_\rho$.

As a coordinate function, we choose a suitable one-dimensional projection $\ell \asymp 0$. Then we define a two-parameter family of functions $u(\alpha,\lambda)$ in $\mathcal{A}_\rho$ by solving

$$(\mathrm{I}-\ell)\mathcal{F}\big(\alpha,u(\alpha,\lambda)\big)=0\,,\qquad \ell u(\alpha,\lambda)=\lambda\hat{u}\,,$$

where $\hat{u}$ is a fixed nonzero function in the range of ℓ. Our goal is to show that for certain rectanges $I\times J$ in parameter space, the equation has a smooth and locally unique solution $u: I\times J\to\mathcal{A}_\rho$. Then locally, the solutions of $\mathcal{F}_\alpha(u)=0$ are determined by the zeros of the function g,

$$g(\alpha,\lambda)\hat{u}=\ell\mathcal{F}\big(\alpha,u(\alpha,\lambda)\big)\,.$$

The equation for $u = u(\alpha, \lambda)$ is equivalent to the fixed point equation for the map $F_{\alpha,\lambda}$, defined by

$$F_{\alpha,\lambda}(u) = (\mathrm{I} - \ell)F_\alpha(u) + \lambda\hat{u}\,.$$

This fixed point problem is solved by converting it to a fixed point problem for a map $\mathcal{C}_{\alpha,\lambda}$, which is obtained from $F_{\alpha,\lambda}$ in the same way that C_α was obtained from F_α. These estimates imply also that $DF_{\alpha,\lambda}(u)$ is invertible, for all u in the ball being considered. Thus, since $F_{\alpha,\lambda}(u)$ is a polynomial in α, λ, and u, the implicit function theorem guarantees that the solution $u = u(\alpha, \lambda)$ depends smoothly on the parameters α and λ.

This leaves the problem of verifying that a certain type of bifurcation occurs. For the sake of definiteness, we will restrict our discussion here to the case of a pitchfork bifurcation. A sufficient set of conditions for the existence of such a bifurcation is given below. A concrete example of a function g that satisfies these conditions (near the origin) is $(\alpha, \lambda) \mapsto \lambda^3 - \alpha\lambda$.

If f is any differentiable function of two variables, denote by $\dot{f}$ and f' the partial derivatives of f with respect to the first and second argument, respectively.

Let $I = [\alpha_1, \alpha_2]$ and $J = [-b, b]$.

Proposition 2. *Let g be a real-valued C^3 function on an open neighborhood of $I \times J$, such that $g(\alpha, 0) = 0$ for all $\alpha \in I$, and*

(1) $g''' > 0 \;\; on \;\; I \times J,$ *(2)* $\dot{g}' < 0 \;\; on \;\; I \times J,$

(3) $g'(\alpha_1, 0) \pm \frac{1}{2} b g''(\alpha_1, 0) > 0,$ *(4)* $\pm g(\alpha_2, \pm b) > 0,$

(5) $g'(\alpha_2, 0) < 0.$

Then $g(\alpha, \lambda) = \lambda G(\alpha, \lambda)$, and the solution set of $G(\alpha, \lambda) = 0$ in $I \times J$ is the graph of a C^2 function $\alpha = a(\lambda)$, defined on a proper subinterval J_0 of J. This function takes the value α_2 at the endpoints of J_0, and satisfies $\alpha_1 < a(\lambda) < \alpha_2$ at all interior points of J_0, which includes the origin.

The role of the computer

The proofs are based on a discretization of the problem, carried out and controlled with the aid of a computer.

At the trivial level of real numbers, the discretization is implemented by using interval arithmetic. In particular, a number $s \in \mathbb{R}$ is “represented” by an interval $S = [S^-, S^+]$ containing s, whose endpoints belong to some finite set of real numbers that are representable on the computer. Such an interval will be called a “standard set” for $\mathbb{R}$.

The collection of all standard sets for $\mathbb{R}$ will be denoted by $\mathrm{std}(\mathbb{R})$. In what follows, a “bound” on a function $g : X \to Y$ is a map G, from a subset D_G of $\mathrm{std}(X)$ to $\mathrm{std}(Y)$, with the property that $g(s)$ belongs to $G(S)$ whenever $s \in S \in D_G$. Bounds on the basic arithmetic operation like $(r, s) \to rs$ are easy to implement on modern computers.

The goal now is to combine these elementary bounds to obtain e.g. a bound G_1 on the norm function on $\mathcal{A}_\rho$, and a bound G_2 on the map $\mathcal{C}$. Then, in order to prove the first inequality it suffices to verify that $G_1(G_2(S)) \subset U$, where S is a set in $\mathrm{std}(\mathcal{A}_\rho^P)$ containing g_0 , and U is an interval in $\mathrm{std}(\mathbb{R})$ with $U^+ < \varepsilon$.

To prove the stability result, we write L_v as a perturbation of a linear operator whose eigenvalues and eigenvectors are known explicitly. Let $\mathcal{T}$ and $\mathcal{L} = \mathcal{T} + A$ be closed linear operators whose spectrum consists of isolated eigenvalues only. Then the following holds.

Proposition 3. *Let Ω be a bounded open subset of $\mathbb{C}$, whose boundary $\partial\Omega$ consists of finitely many rectifiable Jordan curves and avoids the eigenvalues of $\mathcal{T}$. If*

$$\left\| A(\mathcal{T} - z)^{-1} \right\| < 1\,, \qquad \forall z \in \partial\Omega,$$

then $\mathcal{T}$ and $\mathcal{L} = \mathcal{T} + A$ have the same number of eigenvalues (counting multiplicities) in the region Ω, and in its closure.

A proof of this (well known) fact is based on the integral formula

$$P_\Omega = \frac{1}{2\pi i}\int_\Gamma (\mathcal{L} - z)^{-1} dz = \frac{1}{2\pi i}\int_\Gamma (\mathcal{T} - z)^{-1}\left[\mathrm{I} + A(\mathcal{T} - z)^{-1}\right]^{-1} dz$$

for the spectral projection P_Ω of $\mathcal{L}$, associated with the eigenvalues of $\mathcal{L}$ in Ω.

This proposition will be applied not to L_v directly, but to the operator

$$\mathcal{L} = E^{-1} L_v E \,,$$

where $E \asymp \mathrm{I}$ is a suitable linear isomorphism of $\mathcal{A}_\rho$. Here, and in what follows, $U \asymp V$ means that $U - V$ is finite-dimensional.

Corollary 1. *Let $\mathcal{T} \asymp L_0$ and $\mathcal{S} \asymp L_0$ be linear operators on $\mathcal{A}_\rho$, that have no eigenvalues on the imaginary axis. Let $A = \mathcal{L} - \mathcal{T}$. If*

$$\left\|A\mathcal{S}^{-1}\right\| < 1 \,, \qquad \left\|\mathcal{S}(\mathcal{T} - iy)^{-1}\right\| \le 1 \,, \quad \forall y \in \mathbb{R},$$

then L_v and $\mathcal{T}$ have the same number of eigenvalues in the halfplane $\mathrm{Im}(z) > 0$, and in its closure.

In our proof the isomorphism E is chosen in such a way that $\mathcal{L}$ is close to an operator $\mathcal{T}$ that is block-diagonal, in the sense that all eigenvectors of $\mathcal{T}$ are either Fourier monomials (for real eigenvalues) or a linear combination of two Fourier monomials (for pairs of complex conjugate eigenvalues). In order to simplify our estimates, the operator $\mathcal{S}$ is taken to be diagonal. Given $\mathcal{T}$, it is easy to find such an operator $\mathcal{S}$ that satisfies $\|\mathcal{S}(\mathcal{T} - iy)^{-1}\| \leq 1$ without being smaller than necessary.