Rigorous numerics for dissipative Partial Differential Equations II. Periodic orbit for the Kuramoto-Sivashinsky PDE - a computer assisted proof.

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Abstract

We present a method of self-consistent a-priori bounds, which allows to study rigorously dynamics of dissipative PDEs. As an application present a computer assisted proof of an existence of a periodic orbit for the Kuramoto-Sivashinsky equation

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}, \quad u(t,x) = u(t,x+2\pi), \quad u(t,x) = -u(t,-x),$$

where $\nu = 0.127$.

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1 Introduction

The goal of this paper is to develop further the method of self-consistent a-priori bounds for dissipative PDEs. The proposed approach, introduced in [ZM], enables a rigorous computer assisted study of dynamics for dissipative PDEs. The method consists from two basic parts: a reduction of an PDE to a finite dimensional differential inclusion and finite dimensional part, which is based on standard finite dimensional tools from dynamics. While in paper [ZM] the finite dimensional part was based on the Conley index and treated primarily a question of existence of fixed points, in this paper we focus on Poincaré maps and finite-dimensional tools for detection of periodic orbits.

The method of self-consistent bounds is similar in sprit to the Cesari method introduced in [C]. In fact in the context were the Cesari method was used

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originally, namely proofs of existence of solutions to boundary value problem for second order ODEs, our method is slightly stronger, as we drop one of Cesari's conditions (see [ZM, Section 2.4] for more details). We would like to stress here that from our point of view the Cesari method is static, as one can obtain from it only steady states for PDEs.

In our method we look for solutions of dissipative PDE, whose Fourier expansions are converging fast enough, hence are quite regular. This may to appear to be a limitation of the method. But if fact it is not, as solutions which exists for an unbounded time are usually very regular, see [FT, HG, K].

This paper is organized as follows. In Section 2 we present the basics of the method of self-consistent a-priori bounds from [ZM] (with small inessential modifications), which setups the framework for treating all sufficiently high dimensional Galerkin projections as a differential inclusion. Section 3 we discuss the notion of the Poincaré sections and return maps in the context of self-consistent bounds. In Section 4 we recall from [GiZ] the notion of covering relation and we show how it can be used in the context of self-consistent a-priori bounds. In Section 5 we present, as an application of the tools developed in preceding section, a result about an existence of periodic orbit for the Kuramoto-Sivashinsky PDE with odd and periodic boundary conditions for $\nu = 0.127$. This result is formulated as Theorem 18. Section 6 contains a computer assisted proof of this theorem. In Section 7 we describe a modification of the Lohner algorithm [Lo, Lo1, ZLo], which was used in the rigorous integration of differential inclusion in the proof of Theorem 18.

We believe that, besides providing tools for the rigorous detection of periodic orbits for dissipative PDEs, this paper shows that rigorous numerics for dissipative PDEs is possible, hence one can think for example about rigorous numerical shadowing algorithms similar in spirit to the ones developed in [HYC, HYC1, H].

1.1 Notation

Let X be a metric space. Let $U \subset V \subset X$, by $\operatorname{int}_V U$ we will denote an interior of U relatively to V.

Let X be a vector space and $Y \subset X$, then by $\operatorname{conv}(Y)$ we will denote the smallest closed convex set containing Y.

For a set X by 2^X is denote a set of all subsets of X.

For $x \in \mathbb{R}$ we set $|x|_\infty = \max_{i=1,\ldots,n} |x_i|$.

For $S \subset \mathbb{R}^n$ by $\mathrm{IHull}(S)$ we denote an interval enclosure of a set S, i.e. the smallest set of the form $I_1 \times I_2 \times \cdots \times I_n$, where for $i = 1, \ldots, n$, $I_i = [a_i, b_i]$ and $a_i \leq b_i$.

2 The method of self-consistent bounds

In this section we describe the method of self-consistent bounds. The method was introduced in [ZM] and was linked there to the Conley index. In this paper we will show how other finite dimensional topological tools can be used with this method.

Our method begins with the reduction of the full dynamical system to a lower dimensional system which can be studied numerically. In particular, we begin with a nonlinear evolution equation in a Hilbert space H (L^2 in our treatment of Kuramoto-Sivashinsky) of the form

$$\frac{du}{dt} = F(u) \tag{1}$$

where the domain of F is dense in H. By a solution of (1) we understand a function $u : [0, t_{max}) \to \mathrm{dom}\,(H)$, such that u differentiable and (1) is satisfied for all $t \in [0, t_{max})$.

We assume that $\{e_i \mid i = 0, 1, \ldots\}$ forms a complete orthogonal basis for H. In the case of the Kuramoto-Sivashinsky equation $F(u) = Lu + B(u, u)$, where L is the linear part and B is the nonlinear part, the functions $\{e_i\}$ are chosen to be eigenvalues of L.

Fix $m \in \mathbb{N}$ and let

$$P_m : H \to X_m$$

be the orthogonal projection from H onto the finite dimensional subspace spanned by $\{e_1, e_2, \ldots, e_m\}$. Let

$$Q_m := I - P : H \to Y_m$$

be the complementary orthogonal projection. Finally, let

$$A_k : H \to \mathbb{R}$$

be the orthogonal projection onto the subspace generated by e_k.

Given $u \in H$, let $P_m u = p$ and $Q_m u = q$. Equation (1) can be rewritten as

$$\begin{aligned} \frac{dp}{dt} &= P_m F(p, q) && (2) \\ \frac{dq}{dt} &= Q_m F(p, q) && (3) \end{aligned}$$

The strategy adopted is fairly simple: study the dynamics of the low dimensional Galerkin projection (2) to draw conclusions about the dynamics of (1). Before turning to the precise conditions, consider the following heuristic description of our approach.

Let $W \subset X_m$, $V \subset Y_m$ and set $V_j = Q_j(V)$ for $j > m$. Furthermore, given $q_j \in V_j$ assume that $\lim_{j\to\infty} ||q_j|| = 0$. Our only knowledge concerning the higher order modes or “tails” of the solutions is that they project into V. This implies that our knowledge of the vector field is reduced to the following differential inclusion

$$\frac{dp}{dt} \in P_m F(p, V)$$

where $p \in W$. Numerical calculations on this equation are used to find topological invariants (the Conley index, the fixed point index) which guarantee the existence of specific dynamics, e.g. fixed points, periodic orbits, symbolic dynamics, positive entropy, etc. It is simultaneously argued that the topological invariant is the same for any j-dimensional Galerkin projection of (1) for $j \geq m$. Thus, the same dynamical object exists for each sufficiently high Galerkin approximation. Finally, it is shown that the limit of these objects leads to the desired dynamics for the full system (1).

2.1 Self-consistent Bounds

As one might expect the orthonormal basis $\{e_i\}$ and the sets W and V must be chosen with care. The first issue that needs to be dealt with is analytic in nature - solutions to the ordinary differential equations must approximate solutions of the partial differential equation. This leads to the following definition.

Definition 1 Let $m, M \in \mathbb{N}$ with $m \leq M$. A compact set $W \subset X_m$ and a sequence of pairs $\{a_k^\pm \in \mathbb{R} \mid a_k^- < a_k^+,\ k \in \mathbb{N}\}$ form *self-consistent apriori bounds* for (1) if the following conditions are satisfied:

C1 For $k > M$, $a_k^- < 0 < a_k^+$.

C2 Let $\hat{a}_k := \max |a_k^\pm|$ and set $\hat{u} = \sum_{k=0}^{\infty} \hat{a}_k e_k$. Then, $\hat{u} \in H$ and in particular, $\hat{u}$ is bounded in the norm on H ($||\hat{u}|| < \infty$).

C3 The function $u \mapsto F(u)$ is continuous on

$$W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+] \subset H.$$

Moreover, if we define $f_k = \max_{u \in W \oplus \prod_{k=m+1}^{\infty} [a_k^-, a_k^+]} |A_k F(u)|$ and set $\hat{f} = \sum f_k e_k$, then $\hat{f} \in H$. In particular, $||\hat{f}|| < \infty$.

C4a Let $u \in W \oplus \Pi_{k=m+1}^{\infty} [a_k^-, a_k +]$. Then for $k > m$:

$$A_k u = a_k^+ \quad \Rightarrow \quad A_k F(u)) < 0, \tag{4}$$

$$A_k u = a_k^- \quad \Rightarrow \quad A_k F(u)) > 0. \tag{5}$$

The above definition differs slightly from Def. 2.1 in [ZM], namely condition **C4a** is added. In [ZM] this condition was used as part of *topologically self-consistent bounds* (Def. 2.11), which we are not using here.

At this point the reader may wonder if it is hard to find a tail, i.e. $\{a_k^\pm\}$ satisfying **C2-C4a**. This turns out be a relatively easy task. For example, for Kuramoto-Sivashinsky PDEs or Navier-Stokes equations (see [ZM, Z]) to satisfy **C2** and **C3** it is sufficient to take $a_k^\pm = \frac{\pm C}{|k|^s}$ for s large enough. It turns out that with a careful choice of $C = C(W, s)$ condition **C4a** is also satisfied. This choice of tail means that we consider sufficiently regular functions, which may appear a limitation of our approach. But if fact it is not, as solutions which exists for an unbounded time are usually very regular, see [FT, HG, K].

Given self-consistent a-priori bounds W and $\{a_k^{\pm}\}$, let

$$V := \prod_{k=m+1}^{\infty} [a_k^-, a_k^+] \subset Y_m.$$

Our goal is to numerically solve (2) on W and draw conclusions about the dynamics of (1) on the set $W \oplus V \subset H$. To do this we will make use of the following results, first of two of them are obvious:

Lemma 1 *Given self-consistent a priori bounds W and $\{a_k^{\pm}\}$, $W \oplus V$ is a compact subset of H.*

Lemma 2 *Given self-consistent apriori bounds W and $\{a_k^{\pm}\}$, $W \oplus V$, then*

$$\lim_{n\to\infty} Q_n(F(u)) = 0, \quad \text{uniformly for } u \in W \oplus V$$

The following Proposition was proved in [ZM, Prop. 2.4]

Lemma 3 *Let W and $\{a_k^{\pm}\}$ be self-consistent bounds for (1). A function $a : [0,T] \to W \oplus V$ given by*

$$a(t) := \sum_{k=0}^{\infty} a_k(t) e_k$$

is a solution to (1), if and only if, for each $k \in \mathbb{N}$ and all $t \in [0,T]$

$$\frac{da_k}{dt} = A_k F(a). \tag{6}$$

Lemma 4 *Let W and $\{a_k^{\pm}\}$ be self-consistent bounds for (1). Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N} \cup \{\infty\}$ be any sequence. Assume that for all k $x_k : [t_1, t_2] \to W \oplus V$ is a solution of*

$$\frac{dp}{dt} = P_{n_k}(F(p)), \qquad p(t) \in X_{n_k}. \tag{7}$$

Then the family of functions $\{x_k\}$ is relatively compact (i.e. every sequence contains a convergent subsequence).

Proof: From Lemma 1 it follows that set $W \oplus V$ is compact, hence by the Ascoli-Arzela lemma it is enough to show that the functions x_k are equicontinuous.

For this end observe that $|x_k'|$ is bounded from above by $\sup_{x\in W\oplus V} |F(x)|$, which is bounded due to compactness of $W \oplus V$ and condition **C3**. ∎

Lemma 5 *Let W and $\{a_k^{\pm}\}$ be self-consistent bounds for (1). Let $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ be sequence, such that $\lim_{k\to\infty} n_k = \infty$. Assume that for all k $x_k : [t_1, t_2] \to W \oplus V$ is a solution of*

$$\frac{dp}{dt} = P_{n_k}(F(p)), \qquad p(t) \in X_{n_k}. \tag{8}$$

Then there exists a convergent subsequence
$\lim_{l\to\infty} x_{k_l} = x^*$, *where* $x^* : [t_1, t_2] \to W \oplus V$ *and the convergence is uniform on* $[a, b]$. *Moreover,* x^* *satisfies (1).*

Proof: An existence of convergent subsequence follows from Lemma 4. Without any loss of generality we can assume that the whole sequence $\{x_k\}$ converges uniformly to x^*. Obviously x^* is continuous.

We will show that for all $i \in \mathbb{N}$ and $t \in [t_1, t_2]$

$$P_i x^*(t) = P_i x^*(t_1) + \int_{t_1}^{t} P_i(F(x^*(s))\, ds. \tag{9}$$

Let us fix an i. Observe that for k big enough (such that $n_k > i$, hence $P_i P_{n_k} = P_i$) we have

$$P_i x_k(t) = P_i x_k(t_1) + \int_{t_1}^{t} P_i(F(x_k(s))\, ds. \tag{10}$$

Observe that $P_i x_k$ converges uniformly to $P_i x^*$, hence it remains to show that $\int_{t_1}^{t} P_i(F(x_k(s))\, ds$ converges uniformly with respect to $t \in [t_1, t_2]$ to $\int_{t_1}^{t} P_i(F(x^*(s))\, ds$. This follows immediately from the uniform continuity of F on $W \oplus V$, because by **C3** is continuous on $W \oplus V$, hence also uniformly continuous on this set.

From differentiation of (9) we obtain

$$\frac{d}{dt} P_i x^*(t) = P_i F(x^*(t)), \tag{11}$$

hence by Lemma 3 it follows that x^* is a solution of (1). ∎

3 Poincaré sections and maps

In this section we assume **Standing Hypothesis:**
W and $\{a_k^{\pm}\}$ be self-consistent bounds for (1). We set

$$V := \Pi_{k+m+1}^{\infty} [a_k^-, a_k^+]$$

Definition 2 *Let* $\alpha : W \supset dom(\alpha) \to \mathbb{R}$ *be a* C^1*-function defined on some open (with respect to* X_m*) set. We say* $\theta \subset \{x \in (\operatorname{int}_{X_m} W) \oplus V \mid \alpha(P_m x) = 0\}$ *is a* section *for (1) if*

- $P_m(\theta)$ *is* $(m-1)$*-dimensional manifold*
- *there exists a set* U*, such that* $U \subset W$*,* $U = \operatorname{int}_{X_m} U$*,* $\overline{P_m(\theta)} \subset U$ *and a real number* $\beta > 0$ *such that for all* $x \in U$ *and* $q \in V$

$$\left| \sum_{i=1}^{m} \frac{\partial \alpha}{\partial x_k} A_k F(x+q) \right| > \beta$$

Let us make a few comments regarding above definition.

- a section is defined in terms of m-first modes, hence $P_n(\theta)$ is section for n-th Galerkin projection of (1) for $n \geq m$
- the notion of section depends on the self-consistent bounds $W \oplus V$.

The n-dimensional Galerkin projection given by

$$\frac{dx}{dt} = P_n F(x), \qquad x \in X_n \tag{12}$$

induces a local flow $\varphi_n(t, x_0)$, where $\varphi_n(\cdot, x_0)$ is a solution to (12) with an initial condition $x(0) = x_0$.

Definition 3 *Let* θ *be a section for (1). Then* $P_n(\theta)$ *is a section for (12) and we define* a first return time to section θ *function by*

$$t_{n,\theta}(x) = \inf\{t > 0 \mid \varphi_n(t, x_0) \in P_n(\theta)\}, \qquad x \in X_n \tag{13}$$

Definition 4 *Consider sections* θ_1 *and* θ_2*. We assume that either* $\theta_1 = \theta_2$ *or* $\theta_1 \cap \theta_2 = \emptyset$*. Let*

$$D_n = \{x \in P_n(\theta_1) \mid t_{n,\theta_2}(x) < \infty\}. \tag{14}$$

We define a Poincaré map, $G_{n,\theta_1 \to \theta_2} : D_n \to P_n(\theta_2)$*, between sections* θ_1 *and* θ_2 *by*

$$G_{n,\theta_1 \to \theta_2}(x) = \varphi_n(t_{n,\theta_2}(x), x). \tag{15}$$

It is well known, that $G_{n,\theta_1 \to \theta_2}$ *is continuous.*

Now we would like to define a notion of the Poincaré map and that of the first return time for (1). Since we did not assume the local uniqueness for (1) we will only define these notions with respect to a given solution of (1).

Definition 5 *Let* $x : [0, t_{max}) \to H$ *be a solution (1) such that* $x(0) = x_0$*. Let* θ_1*,* θ_2 *be sections, such that either* $\theta_1 = \theta_2$ *or* $\theta_1 \cap \theta_2 = \emptyset$*.*

- $t_{\theta_2}(x_0) = \inf(t > 0 \mid x(t) \in \theta_2)$
- *if* $t_{\theta_2}(x_0) < \infty$ *and* $x_0 \in \theta_1$*, then* $G_{\theta_1 \to \theta_2}(x_0) = x(t_{\theta_2}(x_0))$

Observe that the map Poincaré map G defined above is in principle a multivalued map (it will be in fact multivalued in case of non-uniqueness).

Definition 6 *Let* $\theta_0, \theta_1, \ldots, \theta_r$ *be sections, such that either* $\theta_{i-1} = \theta_i$ *or* $\theta_{i-1} \cap \theta_i = \emptyset$ *for all* $i = 1, \ldots, r$*. We say that* $G_{\theta_{r-1} \to \theta_r} \circ G_{\theta_{r-2} \to \theta_{r-1}} \circ \cdots \circ G_{\theta_1 \to \theta_0}(y_0) = y_r$ *iff and only there exists a solution,* $x : [0, t_{max}) \to H$*, of (1) and a sequence of points* $y_1, \ldots, y_{r-1}$ *such that*

- $y_i \in \theta_i$ *for* $i = 0, \ldots, r$

- *there exists a sequence of real numbers* $0 = t_0 < t_1 < \dots < t_r < t_{max}$*, such that* $x(t_i) = y_i$ *and for all* $t \in (t_{i-1}, t_i)$ $x(t) \notin \theta_i$*.*

Lemma 6 *Let* θ *be a section for (1) and let for* $n \geq m$ $G_n = G_{n,\theta\to\theta}$*.*

Let $\{k_n\} \subset \mathbb{N} \setminus \{1, \dots, m-1\}$*, such that* $\lim_{n\to\infty} k_n = \infty$*. Assume that for all* n *there exists a function* $x_{k_n} : [0, \infty) \to W \oplus P_{k_n}(V)$ *a solution of* $k_n - th$ *Galerkin projection of (1) (i.e. (12) with* $n = k_n$*) , such that* $G_{k_n}(x_{k_n}(0)) = x_{k_n}(0)$*.*

Then there exists a solution $x^* : [0, \infty) \to W \oplus V$ *of (1), such that* $x^*(0) \in \theta$ *and* $G(x^*(0)) = x^*(0)$*. In particular* x^* *is a nonconstant periodic solution of (1).*

Proof: Without any loss of generality we can assume that $k_n = n$ (and we consider only $n > M$).

Let us denote by T_n the period of x_n. Without any loss of generality we can assume $T_n \to T^*$. Observe that from the Def. 2 it follows that there exists $\epsilon > 0$, such that for all n, $T_n \geq \epsilon$, hence also $T^* \geq \epsilon$.

From Lemma 5 it follows that there exists a subsequence x_{n_k} converging uniformly on compact time intervals (we assume that the whole sequence converges) to $x^* : [0, \infty) \to W \oplus V$, which is a solution of (1). Since $x^*(0) \in \theta$, hence x^* is not a constant function. It is also easy to see T^* is a first return time to θ (see Def. 5). It remains to show that $x^*(T^*) = x^*(0)$.

We have

$$\begin{aligned} \|x*(0) - x^*(T)\| \leq \|x^*(0) - x_n(0)\| + \|x_n(0) - x_n(T_n)\| + \\ \|x_n(T_n) - x^*(T_n)\| + \|x^*(T_n) - x^*(T)\| = \\ \|x^*(0) - x_n(0)\| + \|x_n(T_n) - x^*(T_n)\| + \|x^*(T_n) - x^*(T)\|. \end{aligned}$$

First two terms are arbitrarily small as $n \to \infty$ due to uniform convergence and the last term tends to zero due to continuity of x^*. ∎

A slight modification of above proof allows to establish the following

Lemma 7 *Let* θ_i *for* $i = 0, 1, \dots$ *be section for (1), such that either* $\theta_i = \theta_{i+1}$ *or* $\theta_i \cap \theta_{i+1} = \emptyset$ *for* $i = 0, 1, \dots$*.*

Let $S_i = \overline{S_i} \subset P_m\theta_i$*, for* $i \in \mathbb{N}$*.*

Let $\{k_n\} \subset \mathbb{N} \setminus \{1, \dots, m-1\}$*, such that* $\lim_{n\to\infty} k_n = \infty$*. Assume that for all* n *there exists a function* $x_{k_n} : [0, \infty) \to W \oplus P_{k_n}(V)$ *a solution of (12) with* $n = k_n$*, such that for all* $i \in \mathbb{N}$

$$G_{k_n,\theta_i\to\theta_{i-1}} \circ G_{k_n,\theta_{i-2}\to\theta_{i-1}} \circ \dots \circ G_{k_n,\theta_0\to\theta_1}(x_{k_n}(0)) \in S_i.$$

Then there exists a solution $x^* : [0, \infty) \to W \oplus V$ *of (1), such that*

$$G_{\theta_i\to\theta_{i-1}} \circ G_{\theta_{i-2}\to\theta_{i-1}} \circ \dots \circ G_{\theta_0\to\theta_1}(x^*(0)) \in S_i.$$

If for some $r > 0$ *the sequence of sections is* r*-periodic , i.e.* $\theta_i = \theta_{i+r}$ *for* $i \in \mathbb{N}$ *and if additionally for all* n *we have*

$$G_{k_n,\theta_r\to\theta_{r-1}} \circ G_{k_n,\theta_{r-2}\to\theta_{r-1}} \circ \dots \circ G_{k_n,\theta_0\to\theta_1}(x_{k_n}(0)) = x_{k_n}(0)$$

then x^* can be chosen so that x^* is periodic and

$$G_{\theta_r \to \theta_{r-1}} \circ G_{\theta_{r-2} \to \theta_{r-1}} \circ \cdots \circ G_{\theta_0 \to \theta_1}(x^*(0)) = x^*(0).$$

3.1 Basic Differential Inclusion and computation Poincaré maps for all Galerkin projections

In this subsection we discuss how in one step we can obtain an information about Poincaré maps for n-th Galerkin projection for $n > M$.

Consider now a differential inclusion

$$\frac{dx}{dt} \in P_m(F(x)) + E, \qquad x(0) = x_0 \in X_m \tag{16}$$

where $E = \text{conv}\ (\{P_m F(x+q) - P_m F(x) \mid x \in W,\ q \in V\})$ and $x \in C^1$.

We will refer to (16) as a *Basic Differential Inclusion for (1) and self-consistent bounds $W \oplus V$.*

Definition 7 *Consider sections θ_1 and θ_2, such that either $\theta_1 = \theta_2$ or $\theta_1 \cap \theta_2 = \emptyset$. We define a Poincaré map $\mathcal{G}_{\theta_1 \to \theta_2} : \theta_1 \supset dom\,\mathcal{G} \to 2^{\theta_2}$, by*

- *for any $x_0 \in \theta_1$ let $S(x_0)$ be a set of right full solutions of (16), i.e. defined for $t \in [0, t_{max})$, so that either $t_{max} = \infty$ or x cannot be extended to $t > t_{max}$*
- *$x_0 \in dom\,(\mathcal{G}_{\theta_1 \to \theta_2})$ iff for all $x \in S(x_0)$ there exists $t_1 > 0$ such $x(t_1) \in \theta_2$ and*
$$x([0, t_1]) \subset \text{int}_{X_m} W. \tag{17}$$
- *Let $x_0 \in dom\,(\mathcal{G}_{\theta_1 \to \theta_2})$, for $x \in S(x_0)$ let $t_x > 0$ be a smallest positive number with the property $x(t_x) \in \theta_2$. We set*
$$\mathcal{G}_{\theta_1 \to \theta_2}(x_0) = conv\{x(t_x) \mid x \in S(x_0)\},$$

Definition 8 *Given a multivalued map $\mathcal{F} : D \to 2^Y$. A map $f : D' \to Y$ is a selector for F iff $D' \subset D$ and for all $x \in D'$ $f(x) \in F(x)$.*

The following statement is rather obvious, but is of great importance in our approach, hence we alleviate it to the theorem status.

Theorem 8 *Consider sections θ_1 and θ_2, such that either $\theta_1 = \theta_2$ or $\theta_1 \cap \theta_2 = \emptyset$. Let $\mathcal{G} = \mathcal{G}_{\theta_1 \to \theta_2}$ be a Poincaré map for (16). Let $G_n = G_{n,\theta_1 \to \theta_2}$.*

Then for all $n > M$

- *$dom\,\mathcal{G} \oplus P_n V \subset dom\,G_n$*
- *for all $p + q \in dom\,\mathcal{G} \oplus P_n V$ holds*

$$\varphi_n([0, t_{n,\theta}(p+q) \in W \oplus Q_m P_n V$$
$$G_n(p+q) \in \mathcal{G}(p) \oplus \text{int}_{Q_m X_n} Q_m P_n V$$

Proof: Let us consider n-th Galerkin projection

$$\begin{aligned} \frac{dp}{dt} &= P_m F(p+q) \qquad (18) \\ \frac{dq}{dt} &= Q_m P_n F(p+q), \qquad (19) \end{aligned}$$

where $p \in X_m$ and $q \in Q_m X_n$.

Let $p_0 + q_0 \in \operatorname{dom} \mathcal{G} \oplus P_n V$. Let $x(t) = (p(t) + q(t))$ for $t \in [0, t_{max})$ be a solution of system (18-19) with an initial condition $p(0) = p_0$ and $q(0) = q_0$ extended to the right to the maximum existence interval. Observe that $p_0 \in \operatorname{int}_{X_m} W$. From condition **C4a** it follows that $q(t) \in \operatorname{int}_{Q_m X_n} V$ for $t \in (0, h)$ for h sufficiently small.

Observe that as long as $p(t) + q(t) \in \operatorname{int}_{X_m} W \oplus Q_m P_n V$ for all $t \in [0, t_1)$, then $p(t) + q(t)$ for $t \in [0, t_1)$ is a solution of differential inclusion (16). From this observation the assertion follows immediately. ∎

From Theorem 8 it follows that a computation of $\mathcal{G}$ gives bounds for G_n for all $n > M$. The question how to actually compute $\mathcal{G}$ rigorously is treated in [ZPLo], a shorter description can be found in Section 7.

The next theorem, which uses Brouwer fixed point theorem as a finite dimensional tool, illustrates how an information on differential inclusion (16) can be used to obtain a periodic orbit in case of an apparently attracting periodic orbit. A case of unstable orbit is treated in Section 4.4.

Theorem 9 *Assume that we have self-consistent bounds for (1) and a section θ. Let $\mathcal{G}$ be a Poincaré map on θ for (16).*

Assume that there exists a set $B \subset \theta$, such that

- *B is homeomorphic with $(m-1)$-dimensional closed ball,*
- *$B \subset dom\mathcal{G}$,*
- *$\mathcal{G}(B) \subset B$.*

Then there exists a function $u : [0, \infty) \to W \oplus V$ a solution to (1), such that $u(0) \in \theta$ and $G(u(0)) = u(0)$. In particular u is T-periodic for some $T > 0$.

Proof: Consider an n-th Galerkin projection of (1). Let $G_n = G_{n,\theta\to\theta}$. From Theorem 8 it follows that for all $n > M$ we have

$$\begin{aligned} G_n(B \oplus Q_m P_n V) &\subset B \oplus Q_m P_n V \qquad (20) \\ \varphi_n([0, t_{n,\theta}(x_0)], x_0) &\subset W \oplus Q_m P_n V, \quad \text{for } x_0 \in B \oplus Q_m P_n V \qquad (21) \end{aligned}$$

From the Brouwer theorem [DG] it follows that for all $n > M$ there exists a fixed point, x_n, of G_n. Now the assertion follows easily from Lemma 6. ∎

4 Covering relations and self-consistent bounds

In this section we present the notion of *covering relation*, a tool which hopefully will allow to prove an existence of symbolic dynamics for (1). The notion of *a covering relation* was introduced in papers [Z0, Z1, Z2, Z4]. Here we follow the most recent and most general version introduced in [GiZ] and the reader is referred there for proofs.

First we will recall the definition from [GiZ] and then we show how to link it with self-consistent bounds.

4.1 h-sets

Notation: For a given norm in $\mathbb{R}^n$ by $B_n(c,r)$ we will denote an open ball of radius r centered at $c \in \mathbb{R}^n$. When the dimension n is obvious from the context we will drop the subscript n. Let $S^n(c,r) = \partial B_{n+1}(c,r)$, by the symbol S^n we will denote $S^n(0,1)$. We set $\mathbb{R}^0 = \{0\}$, $B_0(0,r) = \{0\}$, $\partial B_0(0,r) = \emptyset$.

For a given set Z, by $\operatorname{int} Z$, $\overline{Z}$, ∂Z we denote the interior, the closure and the boundary of Z, respectively. For the map $h : [0,1] \times Z \to \mathbb{R}^n$ we set $h_t = h(t,\cdot)$. By Id we denote an identity map. For a map f, by $\operatorname{dom}(f)$ we will denote the domain of f.

Definition 9 *A h-set, N, is the object consisting of the following data*

- $|N|$ - *a compact subset of* $\mathbb{R}^n$, *a support of* N
- $u(N), s(N) \in \{0,1,2,\dots\}$, *such that* $u(N) + s(N) = n$
- *a homeomorphism* $c_N : \mathbb{R}^n \to \mathbb{R}^n = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$, *such that*

$$c_N(|N|) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1).$$

We set

$$\begin{aligned} N_c &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1), \\ N_c^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1) \\ N_c^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1) \\ N^- &= c_N^{-1}(N_c^-), \quad N^+ = c_N^{-1}(N_c^+) \end{aligned}$$

Hence a h-set, N, is a product of two closed balls in some coordinate system. The numbers, $u(N)$ and $s(N)$, stand for the dimensions of nominally unstable and stable directions, respectively. The subscript c refers to the new coordinates given by homeomorphism c_N. We will call N^- (N_c^-) an exit set of N and N^+ (N_c^+) an entry set of N. Observe that if $u(N) = 0$, then $N^- = \emptyset$ and if $s(N) = 0$, then $N^+ = \emptyset$.

4.2 Covering relations in finite dimension

For $n > 0$ and a continuous map $f : S^n \to S^n$ by $d(f)$ we denote the degree of f [DG]. For $n = 0$ we define the degree, $d(f)$, as follows. Observe first that $S^0 = \{-1, 1\}$. We set

$$d(f) = \begin{cases} 1, & \text{if } f(1) = 1 \text{ and } f(-1) = -1, \\ -1, & \text{if } f(1) = -1 \text{ and } f(-1) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Definition 10 *Assume* $n > 0$. *Let* $f : \overline{B_n}(0,1) \to \mathbb{R}^n$, *such that*

$$0 \notin f(\partial B(0,1))). \tag{23}$$

We define a map $s_f : S^{n-1} \to S^{n-1}$ *by*

$$s_f(x) = \frac{f(x)}{\|f(x)\|}. \tag{24}$$

Definition 11 *Assume* N, M *are h-sets, such that* $u(N) = u(M) = u$ *and* $s(N) = s(M) = s$. *Let* $f : |N| \to \mathbb{R}^n$ *be continuous. Let* $f_c = c_M \circ f \circ c_N^{-1} : N_c \to \mathbb{R}^u \times \mathbb{R}^s$. *Let* w *be a nonzero integer. We say that*

$$N \overset{f,w}{\Longrightarrow} M$$

(N f*-covers* M *with degree* w*) iff the following conditions are satisfied*

1. *there exists a continuous homotopy* $h : [0,1] \times N_c \to \mathbb{R}^u \times \mathbb{R}^s$, *such that the following conditions hold*

$$\begin{aligned} h_0 &= f_c, && (25)\\ h([0,1], N_c^-) \cap M_c &= \emptyset && (26)\\ h([0,1], N_c) \cap M_c^+ &= \emptyset && (27) \end{aligned}$$

2.1 *If* $u > 0$, *then there exists a map* $A : \mathbb{R}^u \to \mathbb{R}^u$, *such that*

$$\begin{aligned} h_1(p,q) &= (A(p), 0), \quad \text{where } p \in \mathbb{R}^u \text{ and } q \in \mathbb{R}^s && (28)\\ A(\partial B_u(0,1)) &\subset \mathbb{R}^u \setminus \overline{B_u(0,1)} && (29) \end{aligned}$$

Moreover, we require that

$$d(s_A) = w,$$

2.2 *If* $u = 0$, *then*

$$\begin{aligned} h_1(x) &= 0 \qquad \text{for } x \in N_c && (30)\\ w &= 1. && (31) \end{aligned}$$

Intuitively, $N \overset{f}{\Longrightarrow} M$ if f stretches N in the 'nominally unstable' direction, so that its projection onto 'unstable' direction in M covers in a topologically nontrivial manner the projection of M. In the 'nominally stable' direction N is contracted by f. As a result N is mapped across M in the unstable direction, without touching M^+. An example of covering relation on the plane with one unstable direction is shown on Figure 2.

The following theorem was proved in [GiZ]. Various versions of this theorem using slightly weaker notions of covering relations or even without an explicitly defined notion of covering relation were given in [Z0, Z1, Z2, Z4].

Theorem 10 *Assume N_i, $i = 0, \ldots, k$, $N_k = N_0$ are h-sets and for each $i = 1, \ldots, k$ we have*

$$N_{i-1} \overset{f_i, w_i}{\Longrightarrow} N_i \tag{32}$$

Then there exists a point $x \in \operatorname{int} |N_0|$, such that

$$f_i \circ f_{i-1} \circ \cdots \circ f_1(x) \in \operatorname{int} |N_i|, \quad i = 1, \ldots, k \tag{33}$$

Moreover if $N_k = N_0$ then x can be chosen so that

$$f_k \circ f_{k-1} \circ \cdots \circ f_1(x) = x \tag{34}$$

Obviously we cannot make any claim about the uniqueness of x in Theorem 10.

Later we will need the following easy

Lemma 11 *Assume N, M are h-sets, such that $u(N) = u(M) = u$ and $s(N) = s(M) = s$. Let $f : |N| \to \mathbb{R}^n$ be continuous. Assume that there exists a continuous homotopy $h : [0,1] \times |N| \to \mathbb{R}^n$, such that*

$$\begin{aligned} h_0 &= f, && (35) \\ h([0,1], N^-) \cap M &= \emptyset && (36) \\ h([0,1], N) \cap M^+ &= \emptyset && (37) \\ N &\overset{h_1, w}{\Longrightarrow} M, && (38) \end{aligned}$$

then

$$N \overset{f, w}{\Longrightarrow} M$$

4.3 Covering relation with one nominally expanding direction ($u = 1$)

In this section we discuss the case of $u = 1$, hence we have only one nominally expanding and possibly many nominally contracting direction. The basic idea here is: the set N^- consists from two disjoint components and all possible values of the degree w in covering relation are ±1. This allows to give sufficient conditions for an existence of covering relations, which are relatively easy to verify.

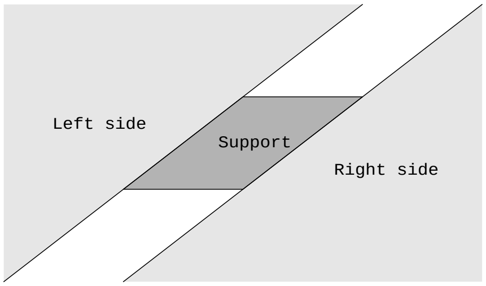

Figure 1: An example of h-set on the plane.

Definition 12 *Let N be a h-set, such that $u(N)=1$. We set*

$$\begin{aligned} N_c^{le} &= \{-1\} \times \overline{B_{s(N)}(0,1)} \\ N_c^{re} &= \{1\} \times \overline{B_{s(N)}(0,1)} \\ S(N)_c^l &= (-\infty,-1) \times \mathbb{R}^{s(N)} \\ S(N)_c^r &= (1,\infty) \times \mathbb{R}^{s(N)}. \end{aligned}$$

We define

$$N^{le} = c_N^{-1}(N_c^{le}), \quad N^{re} = c_N^{-1}(N_c^{re}),$$
$$S(N)^l = c_N^{-1}(S(N)^l), \quad S(N)^r = c_N^{-1}(S(N)^r).$$

We will call N^{le}, N^{re}, $S(N)^l$ and $S(N)^r$ the left edge, the right edge, the left side and right side of N, respectively.

It is easy to see that $N^- = N^{le} \cup N^{re}$.

Remark 12 *A usual picture of a h-set on the plane $u(N)=s(N)=1$ is given in Figure 1.*

A typical picture illustrating covering relation on the plane with one 'unstable' direction is given on Figure 2.

The following theorem was proved in [GiZ].

Theorem 13 *Let N, M be two h-sets in $\mathbb{R}^n$, such that $u(N)=u(M)=1$ and $s(N)=s(M)=s=n-1$. Let $f : |N| \to \mathbb{R}^n$ be continuous.*

Assume that there exists $q_0 \in \overline{B}_s(0,1)$, such that following conditions are satisfied

$$f(c_N([-1,1]\times\{q_0\})) \subset \operatorname{int}(S(M)^l \cup |M| \cup S(M)^r) \tag{39}$$
$$f(|N|) \cap M^+ = \emptyset, \tag{40}$$

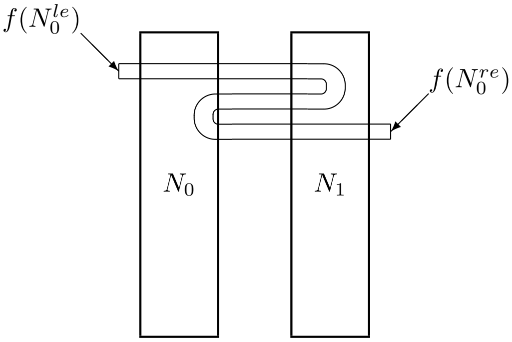

Figure 2: An example of an $f-$covering relation: $N_0 \Longrightarrow N_0, N_1$

and one of the following two conditions holds

$$f(N^{le}) \subset S(M)^l \quad \textit{and} \quad f(N^{re}) \subset S(M)^r \tag{41}$$
$$f(N^{le}) \subset S(M)^r \quad \textit{and} \quad f(N^{re}) \subset S(M)^l. \tag{42}$$

Then there exists $w = \pm 1$, such that

$$N \overset{f,w}{\Longrightarrow} M$$

4.4 Covering relations for self-consistent bounds

In this section we assume that W and $V = \Pi_{k=m+1}^{\infty}[a_k^-, a_k^+]$ form self-consistent bounds for (1). We will use the notation for sections and Poincaré maps introduced in Section 3.

Assume that we have a Poincaré map , $\mathcal{G} = \mathcal{G}_{\theta \to \theta}$, for (1) on section θ. The generalization to Poincaré maps involving transitions between different sections is straightforward.

We want to generalize a notion of an h-set and of a covering relation in the context of self-consistent bounds, so that the Theorem 10 remain valid.

Definition 13 *Let N_0, N_1 be h-sets contained in $\theta \cap W$. Assume that*

$$|N_0| \subset dom \mathcal{G}.$$

We will say that $N_0 \overset{\mathcal{G}}{\Longrightarrow} N_1$ (N_0 $\mathcal{G}$-covers N_1) iff the following conditions are satisfied
there exists a continuous selector, $f : |N_0| \to \mathbb{R}^m$, of $\mathcal{G}$, such that

$$N_0 \overset{f,w}{\Longrightarrow} N_1 \qquad \textit{for some } w \neq 0$$

and

$$
\begin{aligned}
\mathcal{G}(N_0^-) \cap |N_1| &= \emptyset &(43)\\
\mathcal{G}(|N_0|) \cap N_1^+ &= \emptyset &(44)
\end{aligned}
$$

Remark 14 *From the convexity of $\mathcal{G}(x)$ it follows that if $g : |N_0| \to \mathbb{R}^m$ is another selector for $\mathcal{G}$, then by Lemma 11 also $N_0 \overset{g,w}{\Longrightarrow} N_1$ for the same w, which appears in Def. 13.*

Definition 14 *Let N be an h-set contained in $\theta \cap W$. For $k \geq m$ we define an h-set $N(k)$ as follows:*
for $k = m$ we set $N(m) = N$,
for $k > m$ we set

- $|N(k)| = |N| \oplus \Pi_{j=m+1}^{k}[a_j^-, a_j^+]$
- $u(N(k)) = u(N)$, $s(N(k)) = s(N) + (k - m)$
- *Let $e : \mathbb{R}^{k-m} \to \mathbb{R}^{k-m}$ be an affine homeomorphism such that*

  $$e(\Pi_{l=m+1}^{k}[a_l^-, a_l^+]) = \overline{B}_{k-m}(0,1).$$

  We define

  $$c_{N(k)}(x,y) = (c_{N)}(x), e(y)), \qquad x \in X_m, y \in Q_m X_k$$

-
  $$
  \begin{aligned}
  N(k)^- &= N^- \oplus \Pi_{j=m+1}^{k}[a_j^-, a_j^+],\\
  N(k)^+ &= N^+ \oplus \Pi_{j=m+1}^{k}[a_j^-, a_j^+] \cup N \oplus \partial\Pi_{j=m+1}^{k}[a_j^-, a_j^+]
  \end{aligned}
  $$

Observe that in the definition of $N(k)$ we treat the directions e_l for $l > m$ as nominally stable.

We have the following

Lemma 15 *Let N_0, N_1 be h-sets contained in $\theta \cap W$. If $N_0 \overset{\mathcal{G}}{\Longrightarrow} N_1$, then for any $k > M$ $N_0(k) \overset{G_k}{\Longrightarrow} N_1(k)$.*

Proof: Let $a \in H$ be given by $a_k = \frac{a_k^- + a_k^+}{2}$ for $m < k \leq M$ and 0 otherwise. Let f be a selector for $N_0 \overset{\mathcal{G}}{\Longrightarrow} N_1$ (see Def. 13).

We define a homotopy $H : [0,1] \times |N_0(k)| \to X_k$ by

$$H(\lambda, x) = (1-\lambda)G_k(x) + \lambda(f(P_m(x)), a). \tag{45}$$

Obviously $H_0 = G_k$ and $P_m \circ H_1 = f$ for $x \in X_m$. Moreover it is also easy to see that

$$N_0(k) \overset{H_1}{\Longrightarrow} N_1(k). \tag{46}$$

From Theorem 8 it follows that $P_m G_k(x) \in \mathcal{G}(P_m x)$ for all $x \in |N_0(k)|$. By the convexity of $\mathcal{G}(P_m x)$ it follows that also

$$P_m H([0,1],x) \in \mathcal{G}(P_m x). \tag{47}$$

Again from Theorem 8 and the definition of the vector a we know that

$$Q_m H([0,1],x) \in \Pi_{i=m+1}^{n}(a_i^-, a_i^+). \tag{48}$$

From (47), (48), (43) and (44) it follows that

$$\begin{aligned} H([0,1],N_0^-(k)) \cap |N_1(k)| &= \emptyset & (49)\\ H([0,1],|N_0(k)|) \cap N_1(k)^+ &= \emptyset. & (50)\end{aligned}$$

The assertion follows from (46), (49), (50) and Lemma 11. ∎

Now we state the main theorem in this section.

Theorem 16 *Assume N_i, $i = 0,\dots,k$, $N_k = N_0$ are h-sets in $\theta \cap W$ and for each $i = 1,\dots,k$ we have*

$$N_{i-1} \overset{\mathcal{G}}{\Longrightarrow} N_i. \tag{51}$$

Then there exists a point $x \in (\operatorname{int}_{X_m}|N_0|) \oplus V$, such that

$$G^i(x) \in (\operatorname{int}_{X_m}|N_i|) \oplus V, \quad i = 1,\dots,k \tag{52}$$

Moreover, if $N_k = N_0$ then x can be chosen so that

$$G^k(x) = x. \tag{53}$$

Proof: From Lemma 15 it follows that for $n > M$ we have $N_{i-1}(n) \overset{G_n}{\Longrightarrow} N_i$ for $i = 1,\dots,k$. The assertion now follows easily Theorem 10 and Lemmas 6 and 7. ∎

To proceed further we will recall some notions from symbolic dynamics. For any $k > 0$, we define $\Sigma_k^+ = \{0,1,\dots,k-1\}^{\mathbb{N}}$. On Σ_k^+ we define a shift map $\sigma : \Sigma_k^+ \to \Sigma_k^+$ by

$$\sigma((a_0,a_1,a_2,\dots)) = (a_1,a_2,\dots).$$

For a $k \times k$ matrix A, such that $A_{i,j} \in \{0,1\}$ for $i,j = 0,\dots,k-1$ we define $\Sigma_A^+ \subset \Sigma_k^+$ by

$$\Sigma_A^+ = \{(\alpha_i)_{i\in\mathbb{N}} \mid A_{\alpha_{i+1},\alpha_i} \neq 0 \quad \text{for } i = 0,1,\dots\}$$

The following theorem is an immediate consequence of Theorem 16.

Theorem 17 *Let $N_0,\dots,N_{k-1}$ be h-sets in $\theta \cap W$. Let matrix A be given by*

$$A_{ij} = \begin{cases} 1 & \textit{if } N_j \overset{\mathcal{G}}{\Longrightarrow} N_i\ , \\ 0 & \textit{otherwise.} \end{cases} \tag{54}$$

Then for any $\alpha \in \Sigma_A^+$ there exists a solution $x_\alpha : [0,\infty) \to W \oplus V$ so that with respect to this solution $G^i(x_\alpha(0)) \in |N_{\alpha_i}| \oplus V$. Moreover, if α is r-periodic, then x_α can be chosen so that the sequence $G^i(x_\alpha(0))$ is r-periodic, in particular x_α is periodic.

5 The existence of periodic orbit for $\nu = 0.127$

The goal of this section to show the method of self-consistent bounds allows to prove an existence of periodic orbit for the Kuramoto-Sivashinsky partial differential equation.

The Kuramoto-Sivashinsky equation [KT, S] (we will use shortcut KS equation in the sequel) introduced in the context of a wave front propagation is given by

$$u_t = -\nu u_{xxxx} - u_{xx} + 2uu_x \qquad (t,x) \in [0,\infty) \times (-\pi,\pi), \quad \nu > 0. \tag{55}$$

Assuming odd and periodic boundary conditions

$$u(t,x) = u(t,x+2\pi), \quad u(t,x) = -u(t,-x), \tag{56}$$

KS-equation can be reduced (see [ZM]) to the following infinite system of ordinary differential equations for the coefficients of the Fourier expansion of u

$$\dot{a}_k = k^2(1-\nu k^2)a_k - k\sum_{n=1}^{k-1} a_n a_{k-n} + 2k\sum_{n=1}^{\infty} a_n a_{n+k} \quad k = 1,2,3,\ldots \tag{57}$$

We will refer to coordinates a_k as *modes.* In this paper we focus on $\nu = 0.127$. For this parameter value numerical simulations suggests an existence of an attracting limit cycle, whose projection (a_1, a_3) is an ellipse on which the point moves in the clockwise direction.

Theorem 18 *Let $u_0(x) = \sum_{k=1}^{14} -2a_k \sin(kx)$, where a_k are given in Table 1. There exists a function $u^*(t,x)$ a classical solution of (55 - 56) for $\nu = 0.127$, such that*

$$\|u_0 - u^*(0,\cdot)\|_{L_2} < 5 \cdot 10^{-5}, \qquad \|u_0 - u^*(0,\cdot)\|_{C^0} < 7 \cdot 10^{-5} \tag{58}$$

such that u^* is periodic with respect to t, with period $T \in (2.44296, 2.4438)$.

6 Proof of Theorem 18

Let R be a map which leaves even modes and changes sign of odd modes: $a_{2k} \to a_{2k}$ and $a_{2k+1} \to -a_{2k+1}$. It is easy to see that R leaves the system (57) and any Galerkin projection of it invariant (R is the symmetry of the equations).

To define Poincaré sections we use linear functions, i.e. the function α defining the section is linear

$$\alpha(x) = \alpha_0 + \sum_{i=1}^{m} \alpha_i x_i, \quad \alpha' > 0. \tag{59}$$

Table 1: Coordinates of u_0 - an approximation of an initial condition for periodic orbit in Theorem 18.

$a_1 = 2.012106e-01$	$a_2 = 1.289980$
$a_3 = 2.012109e-01$	$a_4 = -3.778662e-01$
$a_5 = -4.230950e-02$	$a_6 = 4.316159e-02$
$a_7 = 6.940217e-03$	$a_8 = -4.156484e-03$
$a_9 = -7.944907e-04$	$a_{10} = 3.316061e-04$
$a_{11} = 7.939456e-05$	$a_{12} = -2.390962e-05$
$a_{13} = -7.087251e-06$	$a_{14} = 1.568377e-06$

Let $\theta_1 = \theta_\alpha$ be such a linear section. We define a symmetric section, θ_2, by $\theta_2 = R\theta_1$. This means that θ_2 is defined by the following linear function β

$$\beta(x) = \alpha_0 + \sum_{i=1}^{m} (-1)^i \alpha_i x_i, \quad \beta' > 0. \tag{60}$$

By the symmetry we have for any Galerkin projection

$$G_{n,\theta_2 \to \theta_1} = R G_{n,\theta_1 \to \theta_2} R \tag{61}$$

For the full Poincaré map on section θ_1, G_n, we obtain that

$$G_n = G_{n,\theta_2 \to \theta_1} G_{n,\theta_1 \to \theta_2} = R G_{n,\theta_1 \to \theta_2} R G_{n,\theta_1 \to \theta_2} = (R G_{n,\theta_1 \to \theta_2})^2. \tag{62}$$

Hence any fixed point for $RG_{n,\theta_1 \to \theta_2}$ is a fixed point for G_n.

Let $\mathcal{G}$ be a Poincaré map for Basic Differential Inclusion (16) for the transition $\theta_1 \to R\theta_1$.

The proof is computer assisted and consists from the following steps, which we will discuss in more details later.

1. an initialization: setting up the parameters: dimensions m and M, finding an approximate periodic orbit, choosing the section θ_1, finding a suitable coordinates on θ_1

2. a construction of self-consistent bounds

3. a construction of a set $N \subset W$, such that $R\mathcal{G}(N) \subset \operatorname{int} N$

4. a conclusion of the proof, application of theorems from previous sections

6.1 Part 1 - initialization

In the proof we used $m = 14$ and $M = 3m = 42$.

We define a point x_0 as in Table 2. This point should be a good approximation to a true periodic point for (55). The point, x_0 was found as follows. Let us

Table 2: Approximate coordinates for the starting point on the periodic orbit for $\nu = 0.127$.

$a_1 = 0.201211$	$a_2 = 1.28998$
$a_3 = 0.201211$	$a_4 = -0.377866$
$a_5 = -0.0423095$	$a_6 = 0.0431616$
$a_7 = 0.00694021$	$a_8 = -0.00415648$
$a_9 = -0.00079449$	$a_{10} = 0.000331606$
$a_{11} = 7.93945e-05$	$a_{12} = -2.39096e-05$
$a_{13} = -7.08724e-06$	$a_{14} = 1.56839e-06$

define a section σ by $a_1 - a_3 = 0$, $a_1' - a_3' > 0$. Consider the map $f = RG_{m,\sigma \to R\sigma}$. Since we are looking for an attractive fixed point we just iterated a map f for some initial value until $|f(x) - x|_\infty < 10^{-6}$. We computed map f using 4-th order Runge-Kutta method with a time step $h = \frac{1}{|2m^2(1-\nu m^2)|} = 0.000106773$.

We define the section θ_1 as a section perpendicular to $P_m F(x_0)$ at x_0, namely we set

$$\alpha(x) = (P_m F(x_0)|x) - (P_m F(x_0)|x_0), \quad \alpha' > 0. \tag{63}$$

Let $\theta_2 = R\theta_1$. We define *section coordinates* on θ_1 , which will be used later in the proof as follows.

The map $g = RG_{m,\theta_1 \to \theta_2} : \theta_1 \supset U \to \theta_1$, where $x_0 \in U$, has x_0 as an approximate fixed point. Next we compute nonrigorously an approximate Jacobian matrix $Dg(x_0)$ using 4-th order Runge-Kutta to compute the trajectory of x_0 and 2-th order Taylor method for the variational part, with the time step $h = \frac{1}{|2m^2(1-\nu m^2)|} = 0.000106773$. The matrix $Dg(x_0) \in \mathbb{R}^{(m-1)\times(m-1)}$ is expressed coordinates $\widetilde{a}_{i=1,\ldots,13}$, defined as follows. Let i_0 be such that $|A_i F(x_0)|$ achieve a maximum for $i = i_0$ (for our periodic orbit $i_0 = 3$). Then

$$\widetilde{a}_i = \begin{cases} a_i & \text{if } i < i_0, \\ a_{i+1} & \text{if } i \geq i_0. \end{cases} \tag{64}$$

Let $r = 8$. From the matrix $Dg(x_0)$ we take an $r \times r$-part upper left corner part, $\widetilde{D} \in \mathbb{R}^{r\times r}$, namely

$$\widetilde{D}_{ij} = Dg(x_0)_{ij}, \qquad \text{for } i,j = 1,\ldots,r \tag{65}$$

Next we apply to $\widetilde{D}$ a diagonalization procedure based on QR-decomposition algorithm [R] to $\widetilde{D}$ to obtain approximate eigenvectors $v_1, v_2, \ldots, v_r$ corresponding to approximate eigenvalues $\lambda_1, \ldots, \lambda_r$. We assume additionally that we have

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|. \tag{66}$$

Parts of diagonalization data are contained in Tables 3 and 4. It is clear from these tables that eigenvalues are decaying rapidly to zero and the high modes are strongly damped.

Table 3: Approximate eigenvalues for $\widetilde{D}$

1	0.5258
2	0.090375
3	3.5019e-08
4	1.6503e-08
5	-3.7784e-09
6	-4.0167e-11
7	-8.9431e-10
8	-6.6955e-11

Table 4: Four leading approximate eigenvectors for $\widetilde{D}$

i	v_1	v_2	v_3	v_4
1	0.31728	-0.070268	0.046791	0.088001
2	-0.81436	0.8341	-0.33709	0.23897
3	0.47674	-0.53787	0.92972	-0.45521
4	0.033933	-0.024953	-0.017497	-0.82473
5	-0.086464	0.095695	-0.13781	0.16446
6	-0.0095825	0.0093288	-0.0094954	0.14042
7	0.01127	-0.012669	0.019939	-0.019339
8	0.0016374	-0.0016399	0.0017784	-0.02442

To an ordered collection of eigenvectors $\{v_1, \dots, v_r\}$ we apply an *interval* Gramm-Schmidt orthogonalization procedure, to obtain a new orthonormal set of vectors $\{w_1, \dots, w_r\}$. These vectors define a new coordinate frame on $\mathbb{R}^r$ and together with coordinates $\widetilde{a}_{r+1}, \dots, \widetilde{a}_m$ define a new coordinates on θ_1, such $x_0 = 0$. We will denote these coordinates by c_i and we will refer to them as *section coordinates*.

It is essential here that the Gramm-Schmidt diagonalization procedure is performed using an interval arithmetics, because in this way we obtain rigorous orthogonal transformation from cartesian coordinates, $\widetilde{a}$, on section θ_1 to *section coordinates* and its rigorous inverse. Both transformations are needed later in the proof, because the integration of Basic Differential Inclusion for (55) is done in cartesian coordinates, but all the important in the proof sets are defined using section coordinates.

6.2 Part 2 - a construction of self-consistent bounds.

We have to define a set $W \subset X_m$ and $\{a_k^{\pm}\}_{k>m}$. In principle it is enough to take any $W \subset X_m$ such that $P_m\gamma \subset W$, where γ is an approximate periodic orbit for m_1-dimensional Galerkin projection of (55) ($m_1 \geq m$). For such a set W we construct $\{a_k^{\pm}\}$ using the algorithm described in Section 3.3 of [ZM] . But it is obvious that to obtain the proof for with a relatively small dimension m (hence in short computation time) it essential that we choose W as a small neighborhood of γ. It turns out also very important how we let $A_k W$ to decay.

The set W was constructed as follows

1. we generate an approximate periodic trajectory for m_1-dimensional Galerkin projection, with $m_1 > m$. In our proof $m_1 = 16$. 4-th Runge-Kutta method with a fixed time step $h = \frac{1}{2|m_1^2(1-\nu m_1^2)|}$ was used for this purpose. As a result of this procedure we obtain a finite ordered set of points $Z \subset X_{m_1}$, which apparently is very close to m_1-dimensional projection of the periodic orbit we are after.

2. we define an auxiliary set $\widetilde{W} \subset X_m$ as follows. On the plane (a_1, a_3) we introduce polar coordinates (r, ϕ). For $i = 0, \dots, p-1$ ($p = 70$ in the proof) we define sets S_i by

$$S_i = \left\{ (a_1, a_3) \mid (a_1, a_3) \neq (0,0), \quad \phi(a_1, a_3) \in \left[\frac{2\pi i}{p}, \frac{2\pi(i+1)}{p} \right] \right\} \tag{67}$$

Next we define sets $D_i(n, Z) \subset X_m$ for $i = 0, 1, \dots, p-1$ by

$$\begin{aligned} D_i(n,Z) &= \mathrm{IHull}\{P_n z \mid z \in Z, \\ &\qquad (A_1(z), A_3(z)) \in S_{(i-1) \bmod p} \cup S_i \cup S_{(i+1) \bmod p}\}. \end{aligned}$$

For an stretching factor parameter, e, ($e = 1.1$ in the proof) and any interval $I = [a, b]$ we define a new interval, $\mathrm{stretch}(I, e)$ by

$$\mathrm{stretch}(I, e) = (a+b)/2 + \left[-e\frac{b-a}{2}, e\frac{b-a}{2} \right]. \tag{68}$$

For an interval set $X = \Pi_{i=1}^{k} I_i$, where I_i is an interval we set

$$\text{stretch}(X, e) = \Pi_{i=1}^{k}\text{stretch}(I_i, e) \tag{69}$$

We introduce another parameter n_{iso} - the number of coordinates for which we force an isolation, but still we include them in computations (in the proof $n_{iso} = 3$).

We define for $i = 0, \ldots, p-1$

$$\widetilde{W}_i = \text{stretch}(D_i(m - n_{iso}), e) \tag{70}$$

We set

$$\widetilde{W} = \sum_{i=0}^{p-1} \widetilde{W}_i. \tag{71}$$

Obviously $\widetilde{W} \subset X_{m-n_{iso}}$. For this set and M we compute the self-consistent bounds to obtain a collection of pairs $\{\widetilde{a}_k^{\pm}\}_{k>m-n_{iso}}$.

We set for $i = 0, \ldots, p-1$

$$W_i = \widetilde{W}_i \times \Pi_{k=m-n_{iso}+1}^{m}[\widetilde{a}_k^{-}, \widetilde{a}_k^{+}] \tag{72}$$

We set $W = \sum_{i=0}^{p-1} W_i$. Table 5 contains the interval enclosure of W i.e. the smallest product of interval containing W.

Finally, for W and M we compute self-consistent bounds $\{a_k^{\pm}\}_{k>m}$ using algorithm outlined in Section 3.3 in [ZM]. Table 6 contains these bounds together with initial bounds used to start the algorithm (see Section 5 in [ZM] for more details).

Let

$$E_n = \text{conv}\left(\{A_n F(x) - A_n(F(P_n(x))) \mid x \in W \oplus \Pi_{k>m}[a_k^{-}, a_k^{+}]\}\right) \tag{73}$$

The interval E_n measures the influence of tail on n-th coordinate of the vector field and we will call the interval vector E *the Galerkin projection error.* Having self-consistent bounds we can compute E in the Basic Differential Inclusion (16). Table 7 contains Galerkin projection errors from the proof.

6.3 Part 3 - Basic Differential Inclusion and the construction of N

Consider now the Basic Differential Inclusion (16) for (55) , where E is computed in previous subsection (see Table 7). Let us remind the reader that by $\mathcal{G}$ we denote a Poincaré map between sections θ_1 and $R\theta_1$. Our goal is to construct a set $N \subset \theta_1 \cap W$, such that

$$\begin{aligned} N &\subset \text{dom}\,(\mathcal{G}) && (74) \\ R\mathcal{G}(N) &\subset \text{int}\, N && (75) \end{aligned}$$

Table 5: The interval enclosure of W from the proof of Theorem 18. The tail is given in Table 6. Columns: c-coordinate index, bounds for c-th coordinate

c	bounds
1	[-0.427501,0.427482]
2	[1.06609,1.33357]
3	[-0.866305,0.866316]
4	[-0.391295,-0.129789]
5	[-0.18118,0.181172]
6	[-0.0271086,0.0474366]
7	[-0.0201397,0.0201411]
8	[-0.00485958,0.00629999]
9	[-0.00182595,0.00182582]
10	[-0.000718572,0.000425729]
11	[-0.000150202,0.000150379]
12	[-3.49005e-05,7.58671e-05]
13	[-1.81432e-05,1.81445e-05]
14	[-6.24288e-06,2.6964e-06]

Let us remind the reader (see condition (17) in Def. 7) that condition (74) requires that any solution of (16) starting from $x \in N$ stays in W for $t > 0$ and less than or equal to the first return time to $R\theta_1$.

The algorithm presented in Section 7 allows to compute rigorous enclosure for $\mathcal{G}(N)$ for any $N \subset \theta_1$.

We constructed N as a result of the following simple algorithm we outline below. We would like to stress that in its description we use *section coordinates* (introduced in Section 6.1).

Algorithm

1. *Initialization* We set the parameters for the computation of $\mathcal{G}$: a time step $h = \frac{1}{2|m^2(1-\nu m^2)|} = 0.000106773$ and $r = 3$ - an order of the numerical method used.

We set $\delta = 2 \cdot 10^{-5}$. We initialize N as follows

$$N = [-\delta, \delta]^{m-1}. \tag{76}$$

2. *Computation of Poincaré map.* We compute $RGN = R \circ \mathcal{G}(N)$ without checking the condition (17), i.e. if the trajectory of N belongs to W. If the computation was terminated successfully then we go to step **3**, otherwise the execution of the algorithm is interrupted and **fail** is returned.

3. If

$$N \subset RGN, \tag{77}$$

then the execution of the algorithm is interrupted and **fail** is returned.

Table 6: The self-consistent bounds from the proof of Theorem 18. $m = 14$, $M = 3m = 42$, initial far tail (for $k > M$) given by $a_k^+ = -a_k^- = 780.898/k^4$. Final far tail (for $k > M$) given by $a_k^+ = -a_k^- = 2.67593e + 10/k^{12}$. Columns form left to right, c-coordinate index, initial bounds for c-th coordinate, final bounds for c-th coordinate (with an isolation for $c > m$), the ratio of diameters of final and initial bounds.

c	initial	end	ratio
15	[-0.0159165,0.0159165]	[-1.98015e-06,1.97985e-06]	0.0001244
16	[-0.0122413,0.0122413]	[-3.16413e-07,5.00703e-07]	3.33754e-05
17	[-0.00957066,0.00957066]	[-1.80681e-07,1.80742e-07]	1.88819e-05
18	[-0.00759164,0.00759164]	[-4.12298e-08,4.03705e-08]	5.37436e-06
19	[-0.00609963,0.00609963]	[-1.48467e-08,1.48436e-08]	2.43379e-06
20	[-0.00495741,0.00495741]	[-4.13123e-09,3.34701e-09]	7.54249e-07
21	[-0.00407088,0.00407088]	[-1.21175e-09,1.212e-09]	2.97694e-07
22	[-0.00337422,0.00337422]	[-2.87756e-10,3.57275e-10]	9.55824e-08
23	[-0.0028206,0.0028206]	[-9.93966e-11,9.93814e-11]	3.52368e-08
24	[-0.00237615,0.00237615]	[-3.01295e-11,2.22132e-11]	1.10142e-08
25	[-0.00201598,0.00201598]	[-8.19686e-12,8.19757e-12]	4.06612e-09
26	[-0.00172161,0.00172161]	[-2.01848e-12,2.23854e-12]	1.23635e-09
27	[-0.00147912,0.00147912]	[-6.76212e-13,6.76347e-13]	4.5722e-10
28	[-0.00127789,0.00127789]	[-1.8801e-13,1.65843e-13]	1.38453e-10
29	[-0.00110977,0.00110977]	[-5.57064e-14,5.5693e-14]	5.01902e-11
30	[-0.000968438,0.000968438]	[-1.46681e-14,1.36154e-14]	1.46027e-11
31	[-0.00084892,0.00084892]	[-4.42188e-15,4.42286e-15]	5.20941e-12
32	[-0.000747297,0.000747297]	[-1.2231e-15,1.23527e-15]	1.64485e-12
33	[-0.000660445,0.000660445]	[-6.11705e-16,6.11627e-16]	9.26143e-13
34	[-0.000585859,0.000585859]	[-7.91392e-16,7.77735e-16]	1.33917e-12
35	[-0.000521518,0.000521518]	[-2.03049e-15,2.0304e-15]	3.89334e-12
36	[-0.000465776,0.000465776]	[-6.03481e-15,6.0351e-15]	1.29568e-11
37	[-0.000417291,0.000417291]	[-1.20708e-14,1.20708e-14]	2.89266e-11
38	[-0.000374957,0.000374957]	[-4.19424e-14,4.19423e-14]	1.1186e-10
39	[-0.00033786,0.00033786]	[-8.8779e-14,8.8779e-14]	2.62769e-10
40	[-0.000305241,0.000305241]	[-1.94964e-13,1.94964e-13]	6.38721e-10
41	[-0.000276468,0.000276468]	[-3.35639e-13,3.35639e-13]	1.21403e-09
42	[-0.000251007,0.000251007]	[-2.9855e-13,2.9855e-13]	1.18941e-09

Table 7: Galerkin errors from the proof of a periodic orbit for $\nu = 0.127$. Computed for $m = 14$, $M = 3m = 42$. The tail is given in Table 6.

k	E_n
1	[-2.69e-11, 2.69036e-11]
2	[-1.57744e-10, 1.5313e-10]
3	[-9.63186e-10, 9.63049e-10]
4	[-2.60273e-09, 2.71426e-09]
5	[-1.49885e-08, 1.49925e-08]
6	[-4.793e-08, 4.63349e-08]
7	[-1.86867e-07, 1.86826e-07]
8	[-6.74122e-07, 6.92668e-07]
9	[-1.75323e-06, 1.75239e-06]
10	[-7.48998e-06, 7.41082e-06]
11	[-1.77666e-05, 1.77637e-05]
12	[-4.47796e-05, 4.37507e-05]
13	[-7.86937e-05, 7.8697e-05]
14	[-3.9583e-05, 4.62362e-05]

If

$$RGN \subset N, \tag{78}$$

then we go to step **4**.

If neither (77) nor (78) is satisfied then we set $N = RGN \cap N$ and jump back to step **2**

4. *Final check.* We recompute $RGN = R \circ \mathcal{G}(N)$, but this time checking condition (17). If this is a case we either return **success** or if we want more tight bounds we jump to step **5** , otherwise **fail** is returned.

5. *Further improvement.* We compute several times: $N = R \circ \mathcal{G}(N)$ and return **success**.

End of algorithm

With computer assistance we proved the following

Lemma 19 *Let $\nu = 0.127$. There exists $N \subset W \cap \theta_1$, N is a product of interval sets in the section coordinates, such that $N \subset dom\mathcal{G}$ and $R \circ \mathcal{G}(N) \subset \operatorname{int} N$.*

About the proof: The main loop was executed three times. Pentium III, 450 MHZ computer was used. A computation of $R\mathcal{G}(N)$ took around 751 seconds.

Table 8 describes the set N, as follows: $N = \Pi_{i=1}^{m-1} N_i$, the interval N_i is given by the i-th row as $N_i = x + r$. For example $N_1 = -3.880465e-07 + [-1.838038e-05, 1.838038e-05]$.

After two iterates we had already an inclusion $R\mathcal{G}(N) \subset N$ (all ratios of diameters in the second column in Table 10 are less than 1). Hence the third

Table 8: The input data from the third iterate of the algorithm from the proof of Theorem 18. $N = x + r$, where r is an interval vector. Columns from left to right are coordinate index, c, x and r. The section coordinates are used for x and r. Output data are in Table 9.

c	x	r
1	-3.880465e-07	[-1.838038e-05,1.838038e-05]
2	-2.115301e-09	[-5.817347e-07,5.817347e-07]
3	4.895745e-10	[-2.728741e-08,2.728741e-08]
4	-2.659308e-10	[-1.082108e-08,1.082108e-08]
5	-2.060927e-10	[-7.368567e-09,7.368567e-09]
6	4.658323e-11	[-6.102844e-09,6.102844e-09]
7	-2.881488e-11	[-5.661919e-09,5.661919e-09]
8	9.706697e-12	[-6.403534e-09,6.403534e-09]
9	4.434868e-10	[-3.128193e-08,3.128193e-08]
10	7.738122e-11	[-1.738133e-08,1.738133e-08]
11	-4.550985e-11	[-2.210128e-08,2.210128e-08]
12	-6.382979e-12	[-2.505782e-08,2.505782e-08]
13	-5.991690e-12	[-1.147062e-08,1.147062e-08]

iterate was used only to improve estimates (see rations is third column in Table 10).

Table 9 contains $R\mathcal{G}(N)$. In Table 10 we illustrate how the set N was changing during the algorithm. Instead of displaying the actual coordinates we display the ratios between the size of the image and the set N in each direction.

6.4 Conclusion of the proof.

From Lemma 19 and an obvious modification of Theorem 9 it follows that there exists a solution, $u^* : [0, T/2] \times (-\pi, \pi) \to \mathbb{R}$ of (55-56) for $\nu = 0.127$ such that

$$R\mathcal{G}_{\theta_1 \to \theta_1}(u^*(0, \cdot)) = R(u^*(T/2, \cdot)) = u^*(0, \cdot). \tag{79}$$

The domain of definition of u^* can be now extended to $[0, \infty) \times (-\pi, \pi)$ using symmetry, R. Observe that the decay rate of the tail of u^* (see Table 6) guarantees that $u^*(t, \cdot)$ is at least a C^{10} function, hence it defines a classical solution of (55).

7 A Lohner-type algorithm for an integration of differential inclusions

In this section we present a Lohner-type algorithm for a rigorous integration of ODEs with a controlled perturbations. This paper relays heavily on [ZLo],

Table 9: The output data from the third iterate of the algorithm from the proof of Theorem 18. $N = x + r$, where r is an interval vector. Columns from left to right are coordinate index, c, $R\mathcal{G}(x + r)$ and $\mathrm{diam}\,(R\mathcal{G}(x + r)$. The section coordinates are used for $R\mathcal{G}(x + r)$. The input data i.e. x and r are given in Table 8.

c	$R\mathcal{G}(x + r)$	$\mathrm{diam}\,(R\mathcal{G}(x + r))$
1	[-1.283714e-05,1.167644e-05]	2.451359e-05
2	[-2.800816e-07,2.784039e-07]	5.584855e-07
3	[-2.350128e-08,2.397283e-08]	4.747412e-08
4	[-9.619064e-09,9.372982e-09]	1.899205e-08
5	[-6.762973e-09,6.567153e-09]	1.333013e-08
6	[-5.891352e-09,5.935270e-09]	1.182663e-08
7	[-5.384946e-09,5.358340e-09]	1.074329e-08
8	[-6.297353e-09,6.307526e-09]	1.260488e-08
9	[-2.359918e-08,2.494372e-08]	4.854291e-08
10	[-1.584270e-08,1.608458e-08]	3.192728e-08
11	[-2.133511e-08,2.120401e-08]	4.253913e-08
12	[-2.457033e-08,2.454699e-08]	4.911733e-08
13	[-1.121413e-08,1.120587e-08]	2.242001e-08

Table 10: Ratios: $\mathrm{diam}\,(A_c R\mathcal{G}(x + r))/\mathrm{diam}\,(A_c r)$ from the proof. Columns from left to right are coordinate index, c, the ratios in first, second and third iteration, i, of the main loop in the algorithm.

c	ratios $i = 1$	ratios $i = 2$	ratios $i = 3$
1	2.759255e+00	9.190188e-01	6.668413e-01
2	1.847428e-01	1.574446e-01	4.800173e-01
3	3.566016e-03	3.826037e-01	8.698905e-01
4	1.376623e-03	3.930301e-01	8.775484e-01
5	8.089586e-04	4.554355e-01	9.045263e-01
6	4.380839e-04	6.965385e-01	9.689436e-01
7	4.862820e-04	5.821643e-01	9.487318e-01
8	4.017088e-04	7.970368e-01	9.842128e-01
9	3.662924e-03	4.270078e-01	7.758937e-01
10	1.312009e-03	6.623943e-01	9.184358e-01
11	1.361729e-03	8.115161e-01	9.623678e-01
12	1.406283e-03	8.909244e-01	9.800796e-01
13	6.549903e-04	8.756328e-01	9.772798e-01

as the proposed algorithm is just a modification running on top of C^0-Lohner algorithm for ODEs described (after [Lo, Lo1]) there.

We study the following ODE

$$x'(t) \quad = \quad f(x(t),y(t)) \tag{80}$$

where $x \in \mathbb{R}^{n_1}$ and $y(t) \in \mathbb{R}^{n_2}$ (we allow for $n_2 = \infty$). Assume that we have some knowledge about $y(t)$, for example $|y(t)| < \epsilon$ for $0 \leq t \leq T$. We would like to find a rigorous enclosure for $x(t)$.

In the context of method self-consistent bounds x in (80) represents a point in X_m ($m = n_1$) and y represents tail. We also know that as long as $x(t) \in W$, then $y(t) \in V$.

7.1 Basic notation

We will use the same conventions as in [ZLo]. In the sequel, by arabic letters we denote single valued objects like vectors, real numbers, matrices. Quite often in this paper we will use square brackets, for example $[r]$, to denote sets. Usually this will be some set constructed in the algorithm. Sets will be also denoted by single letters, for example S, when it is clear from the context that it represents a set. In situations when we want to stress (for example in the detailed description of algorithm) that we have a set in a formula involving both single-valued objects and sets together we will rather use square bracket, hence we prefer to write $[S]$ instead of S to represent a set. From this point of view $[S]$ and S are different symbols in the alphabet used to name variables and formally speaking there is no relation between the set represented by $[S]$ and the object represented by S. Quite often in the description of the algorithm we will have a situation that both variables $[S]$ and S are used simultaneously, then usually $S \in [S]$, but this is always stated explicitly.

For a set $[S]$ by $[S]_I$ we denote the interval hull of $[S]$, i.e. the smallest product of intervals containing $[S]$. The symbol $\mathrm{hull}(x_1,\dots,x_k)$ will denote the interval hull of intervals $x_1,\dots,x_k$. For any interval set $[S]=[S]_I$ by $\mathrm{m}([S])$ we will denote a center point of $[S]_I$. For any interval $[a,b]$ we define a diameter by $\mathrm{diam}([a,b]) = b-a$. For an interval vector or an interval matrix $[S]=[S]_I$ by $\mathrm{diam}\,([S])$ we will denote the maximum of diameters of its components. For an interval $[x^-, x+]$ we set $right([x^-,x^+]) = x^+$ and $left([x^-,x^+]) = x^-$.

For a set $X \subset \mathbb{R}^d$ by $\mathrm{int}\, X$ we denote an interior of X.

For $v,w \in \mathbb{R}^n$ and $A,B \in \mathbb{R}^{n\times n}$ ($n = 1,\dots,\infty$) we say that

$$\begin{array}{lll} v \leq w & \text{iff} & \forall i \quad v_i \leq w_i, \\ A \leq B & \text{iff} & \forall ij \quad A_{ij} \leq B_{ij}. \end{array}$$

7.2 A fundamental estimate

For a fixed $y_c \in \mathbb{R}^{n_2}$ we compare the solutions of two ODEs

$$x' \quad = \quad f(x,y_c), \quad x(t_0) = x_0 \tag{81}$$

$$x' \quad = \quad f(x,y_c) + (f(x,y(t)) - f(x,y_c)), \quad x(t_0) = x_0 \tag{82}$$

where $y(t)$ is the part of solution for the problem (80), which is treated as a known function here.

Let $x_1(t)$ be a solution of (81) and $x_2(t)$ be a solution of (82). We assume that a convex set $[W_y] \subset \mathbb{R}^{n_2}$ is an enclosure for $y([t_0, t_0 + h])$.

Let $[W_1] \subset [W_2] \subset \mathbb{R}^{n_1}$ be convex and compact. We assume that for $s \in [t_0, t_0 + h]$ $x_1(s) \in [W_1] \subset \mathbb{R}^{n_1}$ and $x_2(s) \in [W_2] \subset \mathbb{R}^{n_1}$ for any continuous function $y : [t_0, t_0 + h] \to [W_y]$.

The following lemma is a particular case of Theorem 1 in Section 13 in [W](see subsection IV 'The Lipschitz condition'), a self-contained proof can be also found in [ZPLo].

Lemma 20 *The following inequality holds for $t \in [t_0, t_0 + h]$ and for $i = 1, \ldots, n_1$*

$$|x_{1,i}(t) - x_{2,i}(t)| \leq \left(\int_{t_0}^{t} e^{J(t-s)} C \, ds \right)_i , \tag{83}$$

where

$$\begin{aligned} [\delta] &= \{ f(x, y_c) - f(x, y) \mid x \in [W_1], y \in [W_y] \}, \\ C_i &\geq \sup |[\delta_i]| , \quad i = 1, \ldots, n_1 \\ J_{ij} &\geq \begin{cases} \sup \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) & \textit{if } i = j, \\ \sup \left| \frac{\partial f_i}{\partial x_j}([W_2], [W_y]) \right| & \textit{if } i \neq j. \end{cases} \end{aligned}$$

7.3 One step of the algorithm

Let $\varphi(t, x_0, y_0)$ denotes a solution of equations (80) with an initial condition $x(0) = x_0$ and $y(0) = y_0$. Let $\overline{\varphi}(t, x_0, y_0)$ be a solution of the system

$$x' = f(x, y), \qquad y' = 0 \tag{84}$$

with the same initial conditions $x(0) = x_0$ and $y(0) = y_0$. Observe that for system (84) $y = \text{const}$.

Let $\pi_x : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a projection onto $\mathbb{R}^{n_1}$, i.e. $\pi_x(x, y) = x$.

We are interested in finding rigorous bounds for $\pi_x \phi(t, x, y)$ for $x \in [x_0]$ and $y \in [y_0]$. For this end we propose a modification of the original Lohner algorithm [Lo, Lo1]. Our presentation and notation follows a description of a C^0-Lohner algorithm presented in [ZLo].

In the description below the objects with an index k refer to the current values and those with an index $k + 1$ are the values after the next time step.

One step of the Lohner algorithms is a shift along the trajectory of system (80) with following input and output data:

Input data:

- t_k - a current time,
- h_k - a time step,

- $[x_k] \subset \mathbb{R}^{n_1}$, such that $\pi_x \varphi(t_k, [x_0], [y_0]) \subset [x_k]$,
- $[y_k]$ - bounds for $y(t_k)$.

Output data:

- $t_{k+1} = t_k + h_k$ - a new current time,
- $[x_{k+1}] \subset \mathbb{R}^{n_1}$, such that $\pi_x \varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}]$,
- $[y_{k+1}]$ - bounds for $y(t_{k+1})$.

We do not specify here a form (a representation) of sets $[x_k]$. They can be interval sets, balls, doubletons etc. (see [MZ, ZLo]). This issue is very important in handling the wrapping effect and is discussed in detail in [Lo, Lo1] (see also Section 3 in [ZLo]).

One step of the algorithm consists from the following parts:

1. Generation of a priori bounds for φ. We find a convex and compact set $[W_2] \subset \mathbb{R}^{n_1}$ and a convex set $[W_y] \subset \mathbb{R}^{n_2}$, such that

$$\varphi([0, h_k], [x_k], [y_k]) \subset [W_2] \times [W_y]. \tag{85}$$

2. We fix $y_c \in [W_y]$.

3. Computation of an unperturbed x-projection. We apply one step of the C^0-Lohner algorithm to (84) with a time step h_k and an initial condition given by $[x_k] \times \{y_c\}$. Since $y = \text{const}$ for $\overline{\varphi}$, this is a computation of an ODE in $\mathbb{R}^{n_1}$.

As a result we obtain $[\overline{x}_{k+1}] \subset \mathbb{R}^{n_1}$ and a convex and compact set $[W_1] \subset \mathbb{R}^{n_1}$, such that

$$\begin{array}{rcl} \pi_x \overline{\varphi}(h_k, [x_k], y_c) & \subset & [\overline{x}_{k+1}] \\ \pi_x \overline{\varphi}([0, h_k], [x_k], y_c) & \subset & [W_1] \end{array}$$

4. Computation of perturbation. Using Lemma 20 we find a set $[\Delta] \subset \mathbb{R}^{n_1}$, such that

$$\pi_x \varphi(t_{k+1}, [x_0], [y_0]) \subset \pi_x \overline{\varphi}(h_k, [x_k], y_c) + [\Delta]. \tag{86}$$

Hence

$$\pi_x \varphi(t_{k+1}, [x_0], [y_0]) \subset [x_{k+1}] = [\overline{x}_{k+1}] + [\Delta] \tag{87}$$

5. Computation of $[y_{k+1}]$. This part is not necessary in the bounds for y are known and fixed in advance.

7.4 Part 1 - comments

In the context of a dissipative PDE and self-consistent bounds $W \oplus V$, we set

$$[W_y] = V, \tag{88}$$

and we have to satisfy the following

$$[W_2] \subset W. \tag{89}$$

The last condition is a consistency condition required by Basic Differential Inclusion, namely E is computed under this assumption. In the proof of Theorem 18 in the construction of set N (see Section 6.3) in step 2 we ignore (89), but in the final check (step 4) we need to verify it.

7.5 Part 4 - details

1. We set

$$\begin{array}{rcl} [\delta] & = & [\{f(x,y_c) - f(x,y) \mid x \in [W_1], y \in [W_y]\}]_I \\ C_i & = & \text{right}\left(|[\delta_i]|\right), \quad i = 1,\dots,n_1 \\ J_{ij} & = & \begin{cases} \text{right}\left(\frac{\partial f_i}{\partial x_i}([W_2],[W_y])\right) & \text{if } i = j, \\ \text{right}\left(\left|\frac{\partial f_i}{\partial x_j}([W_2],[W_y])\right|\right). & \text{if } i \neq j. \end{cases} \end{array}$$

   In the context of self-consistent bounds $[\delta]_i = E_i$, where E_i is the Galerkin projection error defined by (73).

2. $D = \int_0^h e^{J(h-s)} C \, ds$
3. $[\Delta_i] = [-D_i, D_i]$, for $i = 1,\dots,n_1$

It remains to explain how we compute $\int_0^t e^{A(t-s)} C \, ds$. First observe that

$$\int_0^t e^{A(t-s)} C \, ds = t \left(\sum_{n=0}^{\infty} \frac{(At)^n}{(n+1)!} \right) \cdot C. \tag{90}$$

We fix any norm $\| \cdot \|$, preferably the L^∞-norm, i.e. $\|x\|_\infty = \max_i |x_i|$ (we should rather choose a norm for which $\|\tilde{A}\|$ is the smallest one and it is easy to compute). Let us set

$$\tilde{A} = At, \qquad A_n = \frac{\tilde{A}^n}{(n+1)!}.$$

In this notation

$$\sum_{n=0}^{\infty} \frac{(At)^n}{(n+1)!} = \sum_{n=0}^{\infty} A_n$$

$$A_0 = \mathrm{Id}, \qquad A_{n+1} = A_n \cdot \frac{\tilde{A}}{n+2}$$

For the remainder term we use the following estimate

$$\|A_{N+k}\| \leq \|A_N\| \cdot \left\| \frac{\tilde{A}}{N+2} \right\|^k$$

Hence if $\left\| \frac{\tilde{A}}{N+2} \right\| < 1$, then

$$\left\| \sum_{n>N} A_n \right\| \leq \|A_N\| \cdot \left\| \frac{\tilde{A}}{N+2} \right\| \cdot \left(1 - \left\| \frac{\tilde{A}}{N+2} \right\| \right)^{-1} \tag{91}$$

7.6 Rearrangement

The rearrangement is an essential ingredient in Lohner algorithm, designed to reduce the wrapping effect [Lo, Lo1, Mo]. We will not discuss this issue here, but we will only include necessary formulas (see [ZLo] for more details and motivation).

Evaluations 2 and 3. In this representation

$$[x_k] = x_k + [B_k][\tilde{r}_k]. \tag{92}$$

In the context of our algorithm in part 3 we obtain

$$[\overline{x}_{k+1}] = \overline{x}_{k+1} + [B_{k+1}][\overline{r}_{k+1}]. \tag{93}$$

Now we have to take into account equation (87). We set

$$x_{k+1} = \mathrm{m}(\overline{x}_{k+1} + [\Delta]) \tag{94}$$

$$[\tilde{r}_{k+1}] = [\overline{r}_{k+1}] + [B_{k+1}^{-1}] \left(\overline{x}_{k+1} + [\Delta] - x_{k+1} \right). \tag{95}$$

Evaluation 4. In this representation

$$[x_k] = x_k + C_k[r_0] + [B_k][\tilde{r}_k]. \tag{96}$$

In the context of our algorithm in part 3 we obtain

$$[\overline{x}_{k+1}] = \overline{x}_{k+1} + C_{k+1}[r_0] + [B_{k+1}][\overline{r}_{k+1}]. \tag{97}$$

Equation (87) is taken into account exactly in the same way as in previous evaluations, i.e. we use equations (94) and (95).

7.7 Computation of the Poincaré map

If as in [ZLo] we assume that the section is given by $\alpha(x) = 0$ then an algorithm discussed in Section 5 in [ZLo] applies also in the present context provided that we have a procedure which gives a rigorous estimates between time step for x-variable in (80). This procedure is described below.

PROCEDURE:
Input parameters:

- h_k - a time step,
- $[x_k] \subset \mathbb{R}^{n_1}$, such that $\pi_x \varphi(t_k, [x_0], [y_0]) \subset [x_k]$,
- $[x_{k+1}] \subset \mathbb{R}^{n_1}$, such that $\pi_x \varphi(t_k + h_k, [x_0], [y_0]) \subset [x_{k+1}]$,
- a convex and compact set $[W_2] \subset \mathbb{R}^{n_1}$ and a convex set $[W_y] \subset \mathbb{R}^{n_2}$, such that
$$\varphi([t_k, t_k + h_k], [x_0], [y_0]) \subset [W_2] \times [W_y], \tag{98}$$
- $y_c \in [W_y]$,
- $[\overline{x}_{k+1}] \subset \mathbb{R}^{n_1}$, such that $\pi_x \overline{\varphi}(h_k, [x_k], y_c) \subset [\overline{x}_{k+1}]$,
- $[W_1] \subset \mathbb{R}^{n_1}$ compact and convex, such that $\pi_x \overline{\varphi}([0, h_k], [x_k], y_c) \subset [W_1]$.

Output:
We compute $[E_k] \subset \mathbb{R}^{n_1}$ such that

$$\pi_x \varphi(t_k + [0, h_k], [x_0], [y_0]) \subset [E_k],$$

Algorithm:

- if $0 \notin f_i([W_2], [W_y])_i$, then i-th coordinate is strictly monotone on $[W_2] \times [W_y]$, hence we set
$$[E_k]_i = \text{hull}([x_k]_i, [x_{k+1}]_i)$$
- if $0 \in f_i([W_2], [W_y])$, then we compute $[\overline{E}_k] \subset \mathbb{R}^{n_1}$, such that
$$\pi_x \overline{\varphi}([0, h_k], [x_k], y_c) \subset [\overline{E}_k] \tag{99}$$
using a procedure for an ODE described in [ZLo]. This procedure requires as input data: h_k, $[x_k]$, $[\overline{x}_{k+1}]$ and $[W_1]$.

  We have
$$\pi_x \varphi(t_k + [0, h_k], [x_0], [y_0])_i \subset [E_k]_i = [\overline{E}_k]_i + [\Delta]_i. \tag{100}$$

A drawback of this approach:
if we have to perform several time steps during which computed enclosure for the trajectory has a nonempty intersection with the section, then Δ is added twice.

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