Estimating Topological Entropy on Surfaces

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December, 2006
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Non-linear dynamical systems exhibit a rich orbit structure. To understand these structures, it is useful to consider various equivalence relations on the class of dynamical systems. A dynamical system (discrete) is a continuous self map $f : X \to X$ where $X$ is a compact metric space. Given an equivalence relation $\sim$ on the class of dynamical systems, the invariants of $\sim$ are the objects which are constant on the equivalence classes. A very useful equivalence relation is

- **Topological Conjugacy:** $f : X \to X$, $g : Y \to Y$ are topologically conjugate if there is a homeomorphism $h : X \to Y$ such that $gh = hf$. Invariants are called dynamical invariants.

We focus on the numerical dynamical invariant called topological entropy — general measure of orbit complexity.
Topological Entropy $h(f)$ of a map $f : X \to X$:

Let $n \in \mathbb{N}$, $x \in X$.

An $n$–orbit $O(x, n)$ is a sequence $x, fx, \ldots, f^{n-1}x$

For $\epsilon > 0$, the $n$–orbits $O(x, n), O(y, n)$ are $\epsilon$–different if there is a $j \in [0, n - 1)$ such that

$$d(f^j x, f^j y) > \epsilon$$

Let $r(n, \epsilon, f) =$ maximum number of $\epsilon$–different $n$–orbits. ($\leq e^{\alpha n} \alpha$)

Set

$$h(\epsilon, f) = \limsup_{n \to \infty} \frac{1}{n} \log r(n, \epsilon, f)$$

(entropy of size $\epsilon$)

and

$$h(f) = \lim_{n \to \infty} h(\epsilon, f) = \sup_{\epsilon > 0} h(\epsilon, f)$$

(topological entropy of $f$) [$\epsilon$ small $\Rightarrow$ $f$ has $\sim e^{h(f)n}$ $\epsilon$– different orbits]
Properties of Topological Entropy

- **Dynamical Invariant:** $f \sim g \implies h(f) = h(g)$
- **Monotonicity of sets and maps:**
  - $\Lambda \subset X, f(\Lambda) \subset \Lambda \implies h(f,\Lambda) \leq h(f)$
  - $(g, Y)$ a factor of $f$: $\exists \pi : X \to Y$ with $gh = hf \implies h(f) \geq h(g)$
- **Power property:** $h(f^n) = nh(f)$ for $N \in \mathbb{N}$.
- **$h : \mathcal{D}^\infty(M^2) \to \mathbb{R}$ is continuous (in general usc for $C^\infty$ maps)**
- **Variational Principle:**
  $$h(f) = \sup_{\mu \in \mathcal{M}(f)} h_\mu(f)$$
Examples of Calculation of Topological Entropy

Topological Markov Chains TMC (subshifts of finite type SFT)
First, the full $N$–shift:
Let $J = \{1, \ldots, N\}$ be the first $N$ integers, and let

$$\Sigma_N = J^\mathbb{Z} = \{a = (\ldots, a_{-1} a_0 a_1 \ldots), \ a_i \in J\}$$

with metric

$$d(a, b) = \sum_{i \in \mathbb{Z}} \left| \frac{a_i - b_i}{2^{\mid i \mid}} \right|$$

This is a compact zero dimensional space (homeomorphic to a Cantor set)
Define the left shift by

$$\sigma(a)_i = a_{i+1}$$

This is a homeomorphism and $h(\sigma) = \log N$. 
Let $A$ be an $N \times N$ 0-1 matrix and consider

$$\Sigma_A = \{a \in \Sigma_N : A_{a_i a_{i+1}} = 1 \ \forall i\}$$

Then, $\sigma(\Sigma_A) = \Sigma_A$ and $(\sigma, \Sigma_A)$ is a TMC.

One has

$$h(\sigma, \Sigma_A) = \log \text{sp}(A) \quad (\text{sp}(A) : \text{spectral radius of } A)$$

**Definition.** A subshift of $f$ is an invariant subset $\Lambda$ such that $(f, \Lambda) \sim (\sigma, \Sigma_A)$ for some 0-1 matrix $A$.

**Theorem. (Katok)** Let $f : M^2 \to M^2$ be a $C^2$ diffeomorphism of a compact surface with $h(f) > 0$. Then,

$$h(f) = \sup_{\text{subshifts } \Lambda \text{ of } f} h(f, \Lambda).$$

So, to estimate entropy on surfaces, we should look for subshifts.
Hyperbolic Fixed Points, Stable and Unstable Manifolds

Let $M = M^2$ be a smooth surface, and let $\mathcal{D}(M)$ denote the space of $C^\infty$ diffeomorphisms from $M$ to $M$. Give $M$ a Riemannian metric with associated distance $d$.

Let $f \in \mathcal{D}(M)$, and let $p$ be a hyperbolic fixed point (i.e., $f(p) = p$, eigenvalues of $Df(x)$ have norm $\neq 1$)

Let $\lambda_u, \lambda_s$ denote the eigenvalues of $Df_p$ with $|\lambda_u| > 1, |\lambda_s| < 1$.

Let $T_pM = E^u \oplus E^s$ be the associated eigenspaces.

Let

$$W^s(p) = \{y \in M : d(f^n y, f^n x) \to 0 \text{ as } n \to \infty\}$$

$$W^u(p) = \{y \in M : d(f^{-n} y, f^{-n} x) \to 0 \text{ as } n \to \infty\}$$

Then, $W^u(p), W^s(p)$ are injectively immersed ($C^\infty$) curves tangent at $p$ to $E^u(p), E^s(p)$, respectively.

(Analogous results for hyperbolic periodic points $p$ with $f^T(p) = p$)

Set

$$W^\nu(O(p)) = \bigcup_{z \in O(p)} W^\nu(z) \text{ for } \nu = s, u.$$
Homoclinic Points and Homoclinic Tangles

Let $p$ be a hyperbolic periodic point with orbit $O(p)$. A point $q \in (W^u(O(p)) \setminus O(p)) \cap W^s(O(p))$ is called a homoclinic point. It is transverse if the curves $W^u(O(p))$ and $W^s(O(p))$ are not tangent at $q$.

Fact: (Katok) $f$ has Transverse homoclinic points iff $f$ has subshifts iff $h(f) > 0$

Definition. Homoclinic Tangle = compact set which is the closure of the transverse homoclinic points of a hyperbolic periodic orbit.

Fact: A homoclinic tangle is an $f$–invariant set with a dense orbit and a dense set of hyperbolic periodic orbits.

Using results of Katok-Yomdin-SN get:

$f \in D^\infty(M^2), h(f) > 0, M^2$ compact $\implies$ there is a homoclinic tangle $\Lambda$ such that $h(f) = h(f, \Lambda)$. 
Typical picture of a homoclinic tangle
Consider the Henon family \( H(x, y) = (1 + y - a \times x^2, b \times x) \)
**Standard Henon Map:** \( a = 1.4, b = 0.3 \)

**Figure:** Homoclinic tangle for Henon map Stable and Unstable manifolds computed with Dynamics-2 (Nusse, Yorke)
Can one estimate entropy using homoclinic points? —Yes. To illustrate: Consider the standard geometry associated to the Smale horseshoe diffeomorphism $f$.

The set $\bigcap_n f^n(Q) = \Lambda$ is such that $(f, \Lambda) \sim (\sigma, \Sigma_2)$. So, $h(T) = \log 2$.

In general, if one sees the geometry of the horseshoe in a map $f$, then $h(f) \geq \log 2$. 
As an example, using a result of K. Burns and H. Weiss, and the program Dynamics 2 of Nusse and Yorke, can easily see how to get

\[ h(H) > \frac{1}{2} \log(2) = 0.34657 \]

Pieces of \( W^u, W^s \) of right fixed point, a quadrilateral, and its second image

**Figure:** Quadrilateral computed with Dynamics-2
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Pieces of \( W^u, W^s \) of right fixed point, a quadrilateral, and its second image.
Remarks.

- Double precision floating point accuracy: $\approx 10^{-16}$
  Graphics resolution (i.e., pixel size $\approx 10^{-3}$),
  So, can prove by hand (or with computer) that 2nd images of quadrilateral look as in pictures.

- for better estimation of entropy would need much finer methods.

- Systematic Method: rigorously compute long pieces of pieces of stable and unstable manifolds and use them to construct subshifts —use of trellises

- We describe trellises. For rigorous numerical implementation, see the talk of J. Grote
Some previous work on numerical Estimation of entropy in the Henon family

\( h(H) > 0 \) simply from transverse homoclinic points

Existence of transverse homoclinic points

- Misiurewicz-Szewc, (by hand)
- Francescini-Russo (computer-assisted, parametrizations of stable and unstable manifolds, later used by Gavosto-Fornaess for quadratic tangencies)

Interval arithmetic:

- Stoffer-Palmer (1999)- \( H^{25} \) has a full 2-shift via rigorous shadowing, (Note: Later, we show \( H^2 \) has a 2-shift factor)
- Galias-Zgliczynski (2001): specific subshifts geometrically via interval bounds, best lower bound: \( h(H) > 0.430 \), via subshift-29 symbols
- attempts to estimate \( N_n(H) \) –up to all periodic points of order 30. in hyperbolic systems, \( h(f) = \limsup_{n \to \infty} \frac{1}{n} \log N_n(f) \)
Figure 2: Enclosure of the nonwandering part of $[-1, 2] \times [-2, 2]$ dynamics is defined. Since the nonwandering part is composed of 8 connected subsets, we choose 8 quadrangles (see Fig. 3(a)). There are only four covering relations between these sets. The transition matrix is almost empty and hence there is no interesting symbolic dynamics on these sets. We modify the position of the rectangles by hand, so that a large number of covering relations hold. The improved sets and their images under the Hénon map are shown in Fig. 3(b).

Finally, we check rigorously the existence of covering relations between the chosen sets. The coverings correspond to the symbolic dynamics on eight symbols with the following transition matrix:

$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
$$

(7)

It follows that the symbolic dynamics with the transition matrix (7) is embedded in $\mathcal{h}$ and that the topological entropy of the Hénon map is bounded by $H(\mathcal{h}) > 0.382$. This is better than the best estimate known to date ($H(\mathcal{h}) > 0.338$, see [3]).

We have performed several other attempts to find complex symbolic dynamics for the Hénon map. The largest bound for the topological entropy $H(\mathcal{h}) > 0.430$ was obtained for the sets shown in Fig. 3(c). This bound is close to the non-rigorous estimation of topological entropy based on the number of low-period cycles $H(\mathcal{h}) \approx 0.465$ (see [2]).

Figure 3: (a) Symbolic dynamics on 8 symbols, initial quadrangles, (b) Symbolic dynamics on 8 symbols, improved quadrangles, (c) Symbolic dynamics on 29 symbols

Figure: Galias Subshift with $h(H) > 0.430$, 29 symbols
Table 7. Periodic orbits for the Hénon map belonging to the trapping region. $Q_n$, number of periodic orbits with period $n$; $P_n$, number of fixed points of $h^n$; $H_n(h) = n^{-1} \log(P_n)$, estimation of topological entropy based on $P_n$.

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<th>$n$</th>
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Figure: Galias Periodic Table
Trellises and Associated Subshifts.

Let \( f : M \to M \) be a smooth surface diffeomorphism

Let \( P \) be finite invariant set of hyperbolic saddle orbits with associated stable and unstable manifolds \( W^u(p), W^s(p), p \in P \)

For each \( p \in P \), let \( W_1^u(p) \subset W^u(p), W_1^s(p) \subset W^s(p) \) be a compact, connected relative neighborhoods of \( p \) in \( W^u(p), W^s(p) \), resp.

Set \( T^u = \bigcup_{p \in P} W_1^u(p), T^s = \bigcup_{p \in P} W_1^s(p) \)

The pair \( T = (T^u, T^s) \) is a Trellis if \( f(T^u) \supset T^u, f(T^s) \subset T^s \)

An associated rectangle \( R \) for the trellis \( T = (T^u, T^s) \) is the closure of a component of the complement of \( T^u \cup T^s \) whose boundary is a Jordan curve which is an ordered union of exactly four curves \( C_1^u, C_2^s, C_3^u, C_3^s \) with \( C_i^u \subset T^u, C_i^s \subset T^s \).

Set \( \partial^u(R) \stackrel{\text{def}}{=} C_1^u \cup C_3^u, \partial^s(R) \stackrel{\text{def}}{=} C_2^s \cup C_4^s \)
Trellises: studied by R. Easton, Garrett Birkhoff
Pieter Collins: Studied relation to Bestvina-Handel, Franks-Misiurewicz methods for forcing orbits and isotopy classes mod certain periodic orbits
For a rectangle $R$ with $\partial^u(R) = C_1^u \cup C_3^u$, $\partial^s(R) = C_2^s \cup C_4^s$, define an $R$–$u$-disk = topological closed 2-disk $D$ with $\text{int}(D) \subset R$, $\partial D \subset W^u(p) \cup W^s(p)$, and $\partial D$ meeting both parts of $\partial^s(R)$. An $R$–$s$-disk in $R =$ topological closed 2-disk $D$ with $\text{int}(D) \subset R$, $\partial D \subset W^u(p) \cup W^s(p)$, and $\partial D$ meeting both parts of $\partial^u(R)$.

Figure: u-disk
Given a Trellis $T$, we obtain a SFT as follows.
Let $\mathcal{R}(T)$ denote the collection of all associated rectangles:

$$\mathcal{R}(T) = \{R_1, R_2, \ldots, R_s\}$$

We say that $R_i \prec_f R_j$ if

- $f(R_i) \cap R_j$ contains an $R_j$–u-disk, and
- $R_i \cap f^{-1}(R_j)$ contains an $R_i$–s-disk.

Define the incidence matrix $A$ of the trellis $T = 0$-1 matrix such that $A_{ij} = 1$ iff $R_i \prec R_j$. Set $(\sigma, \Sigma_A) = \text{associated SFT}$.

**Theorem** Let $T$ be a trellis for $C^\infty$ surface diffeomorphism $f$ with associated SFT $(\sigma, \Sigma_A)$. Then,

$$h(f) \geq h(\sigma, \Sigma_A).$$
• Idea of Proof: If $R_i \prec_f R_j$ and $R_j \prec_f R_k$, then $R_i \prec_{f^2} R_k$.

In a word $R_{i_0} R_{i_1} \ldots R_{i_k}$ of $R'_i$s, get pieces of disjoint parts of $\partial^u(R_i)$ whose $f^k$—images stretch across $R_{i_k}$.

So, get curves whose length growth $\geq h(\sigma, \Sigma_A)$.

• Remark. Since $R'_i$s not disjoint, may not have $(\sigma, \Sigma_A)$ as a factor.

May have other SFT’s with entropy near $h(\sigma, \Sigma_A)$ as factors.

Remark. Given rectangles associated with a trellis, we can consider subcollections of them and first return maps to induce various SFT’s which give lower bounds for entropy.

Next, we consider some good pieces of $W^u(p)$, $W^s(p)$ for estimation of $h(H)$.
Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

Figure: 7th backward iterate of stable manifold
Some good trellises

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Rigorous computation of stable and unstable manifolds with COSY.

**Figure:** 8th backward iterate of stable manifold
Some good trellises

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Rigorous computation of stable and unstable manifolds with COSY.

**Figure:** 9th backward iterate of stable manifold
Some good trellises

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Rigorous computation of stable and unstable manifolds with COSY.

![Diagram](image)

**Figure:** 10th backward iterate of stable manifold
Some good trellises

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Figure: 11th backward iterate of stable manifold
joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

Figure: 12th backward iterate of stable manifold
Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

Figure: 13th backward interate of stable manifold
Some good trellises

joint with M. Berz, K. Makino, J. Grote (Phys, MSU)
Rigorous computation of stable and unstable manifolds with COSY.

Figure: with longer piece of $W^u$
Let
\[ p \approx (0.6313544770895048, .1894063431268514) \]
be the right fixed point of
\[ H(x, y) = (1 + y − 1.4 * x^2, 0.3 * x) \]
Let \( T = (T^u, T^s) \) be the "first trellis" of \( H^2 \): i.e., "D" shaped trellis containing \( p \) for \( H^2 \).
Using rectangles obtained from the piece of \( T^u \) and \( H^j T^s, 0 \leq j \leq 11 \), we constructed a 42x42 matrix \( A \) whose entries are 0’s, 1’s, 2’s which corresponds to a "SFT" in \( H \).
This means that refining \( A \) to an incidence matrix \( A_1 \) (i.e., getting rid of the 2’s), gives a trellis and associated SFT \((\sigma, \Sigma A_1)\) with entropy
\[ h(H) \geq h(\sigma, \Sigma A_1) \approx 0.4563505671076695 \approx 0.456 \]
Here the \( \approx \) means up to the calculation of the spectral radius of \( A_1 \) (done using maxima).
Comments on Numerical Methods for Computing Invariant Manifolds

- Graph Transform not generally used: have formula $f_2(1, g) \circ [f_1(1, g)]^{-1}$. So, need to do an inversion.
- Parametrization Method: Francescini-Russo, Gavosto-Fornaess, J. Hubbard, Carré, Fontich, de la Llave.
  Justification: use power series methods, truncate, and get estimates of remainders
- Bisection Method, like a newton method, completely rigorous, not really used in most programs

Remark Using shadowing ideas and volume estimates, all of these can be made rigorous in the $C^0$ (i.e., enclosure) sense.