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The method of topological sections in the rigorous numerics of dynamical systems

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Consider the following differential equation in the complex plane
\[ z' = (1 + e^{i\varphi t}|z|^2)\overline{z}. \]

**Theorem.** (Srzednicki, Wójcik 1997)
For \( \varphi \in (0, 1/288] \) the Poincaré map of this equation admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on two symbols.

**Theorem.** (Wójcik, Zgliczyński, 2000)
For \( \varphi \in (0, 495/1000] \) the Poincaré map of this equation admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on three symbols.
Adding the equation
\[ t' = 1 \]
we obtain an ODE which induces a flow on \( \mathbb{R} \times \mathbb{C} \).

• The right-hand-side of the equation is \( T \) periodic in \( t \) variable with \( T = 2\pi/\varphi \).

• Therefore there is an induced flow on \( S^1_T \times \mathbb{C} \), where \( S^1_T = [0, T]/\sim \) with \( \sim \) the relation identifying \( 0 \) and \( T \).

• One studies the Poincaré map \( P \) on the Poincaré section \( X := \{0\} \times \mathbb{C} \).

• The dynamical features of this Poincaré map may be captured by means of so called isolating segments.
Poincaré map
Poincaré map
Poincaré map
Comparing the two proofs one can guess that the analytical complexity of the proof grows as $\varphi$ grows.

Question: Is it possible to provide a computer assisted proof, for instance for $\varphi = 1$?

• Bad news: it is difficult to find useful algorithms constructing isolating blocks for flows
• Good news: The topological criterion used in the Srzednicki-Wójcik proof does have a counterpart for maps
Let $f : X \to X$ be a map and let $N \subset X$ be compact. The set $N$ is an isolating neighborhood if

$$\{ x \in N \mid \forall n \in \mathbb{Z} \ f^n(x) \in N \} \subset \text{int} \ N.$$  

A pair of compact $P = (P_1, P_2)$ subsets of $N$ is an index pair if

- $x \in P_i, f(x) \in N \implies f(x) \in P_i, \ i = 1, 2$
- $x \in P_1, f(x) \not\in N \implies x \in P_2$

$\text{Inv} N \subset \text{int}(P_1 \setminus P_2)$. 

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The associated \textit{indexed} map is
\[ I_P := H^*(f_P) \circ H^*(i_P)^{-1} : H^*(P_1, P_2) \to H^*(P_1, P_2) \]
where
\[ f_P : (P_1, P_2) \ni x \to f(x) \in (P_1 \cup f(P_2), P_2 \cup f(P_2)) \]
\[ i_P : (P_1, P_2) \ni x \to x \in (P_1 \cup f(P_2), P_2 \cup f(P_2)) \]
The \textbf{generalized kernel} of \( I_P \) is
\[ \text{gker}(I_P) := \bigcup_{n \in \mathbb{N}} \ker I_P^n. \]
The \textbf{Conley index} is
\[ (CH^*(S, f), \chi(S, f)) := (H^*(P_1, P_2)/\text{gker}(I_P), [I_P]). \]
G-horseshoe example
Let $X$ be an ENR. Assume that $M \subset N$ are isolating blocks with respect to $f$ such that

(a): $\chi_M = \text{id}_Q$, $\chi_N = -\text{id}_Q$,

(b): $f(N) \cap M \cap f^{-1}(N) \subset \text{int}(M)$,

(c): $f(N \setminus f^{-1}(\text{int}(M))) \cap M \subset N^-$,

(d): all inclusions in the diagram

\[
\begin{array}{ccc}
(M, M^-) & \longrightarrow & (N, N \setminus f^{-1}(\text{int}(M))) \\
\downarrow & & \downarrow \\
(M, M \cap N^-) & \longrightarrow & (N, N^-)
\end{array}
\]

induce isomorphisms in the Alexander-Spanier cohomology.
Put $I = \text{inv}_f N = \text{inv}_f (\text{cl}(N \setminus N^-))$. Let $\Sigma_2 = \{0, 1\}^\mathbb{Z}$ and $\sigma : \Sigma_2 \to \Sigma_2$ be a shift map.

**Theorem.** (K. Wójcik, MM, 2003)
There is a continuous, surjective map $g : I \to \Sigma_2$ such that $f$ restricted to $I$ is semiconjugated by $g$ to the shift $\sigma$ i.e. $g \circ f = \sigma \circ g$. Moreover, for any $n$-periodic sequence of symbols $c \in \Sigma_2$ its counterimage $g^{-1}(c)$ contains an $n$-periodic point for $f$. 

⊙
A discrete analog of Srzednicki’s criterion
Problem: extremely strong expansion

- Almost every trajectory of this equation escapes in a short time to infinity.
- The expansion of the Poincaré map is extremely strong.

- Escape time computation
- Expansion
Intermediate sections

- intermediate sections $\Rightarrow$ compose intermediate multivalued maps to get the resulting multivalued enclosure of the Poincaré map
- intermediate topological sections $\Rightarrow$ find the index map from section to section and compose maps in homology
Assume $0 < a < t_1 < t_2 < \ldots < t_{n-1} < t_n = T$ and $R > 3$. Put

\[ X_i : = \left[-R, R\right] \times \left[-R, -R\right] \times \left[a, t_i\right] \cup \left[R, R\right] \times \left[-R, -R\right] \times \left[a, t_i\right] \cup \left[-R, -R\right] \times \left[-R, R\right] \times \left[a, t_i\right] \cup \left[-R, R\right] \times \left[R, -R\right] \times \left[t_n, t_i\right] \]

and $X_0 := X_n$. \(\Box\)
For $i = 2, 3, \ldots, n$ we have well defined Poincaré maps

$$f_i : X_{i-1} \to X_i$$

and for some small $\epsilon > 0$ the Poincaré map

$$f_0 : [-R + \epsilon, R - \epsilon] \times [-R + \epsilon, R - \epsilon] \times [0, 0] \to X_1$$
Proof of continuity
Define

\[ X := \bigcup_{i=1}^{n} X_n \]
\[ f := \bigcup_{i=1}^{n} f_n \]

For an isolating neighborhood \( N \subset X \) the index map \( \chi \) decomposes as

\[ \chi = \chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_n. \]

It turns out that the requested index map of the Poincaré map is

\[ \chi_n \circ \chi_{n-1} \circ \cdots \circ \chi_1. \]

- Computation of the \( \text{id} \) index map
- Computation of the \(-\text{id}\) index map
- \(-\text{id}\) map - section 0
- \(-\text{id}\) map - section 20
- \(-\text{id}\) map - section 40
Theorem. (MM 2004)
For $\varphi = 1$ the Poincaré map of the equation
\[ z' = (1 + e^{i\varphi t}|z|^2)\bar{z}. \]
admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on two symbols.
It is possible to get rid of intermediate sections entirely and get the required index map directly from the index pair for the flow.

**Theorem.** (MM, R. Srzednicki, 2005)

Assume \((W, W^*) \subset \mathbb{R} \times \mathbb{C}\) is an isolating segment over \([0, T]\).

Let \(c \in C_q(W)\) be such that

\[
\partial c = c_0 + c^- + c_T
\]

for some \(c_0 \in Z_{q-1}(W_0, W^*_0), c_T \in Z_{q-1}(W_T, W^*_T)\) and \(c^- \in C'_{q-1}(W^*)\). Then

\[
\mu_W([c_0]) = [c_T].
\]
The theorem shows that to find the Conley index of the Poincaré map it is enough to:

- find a candidate for an isolating segment
- verify isolation
- find a sufficiently large subset of the exit set, so that the chains \( c \) in the above theorem may be constructed for all homology generators in \( H_*(W_0, W_0^*) \).

There is no need to find the whole exit set.
New developments
For $\varphi \in [0.495, 0.5] \cup [0.997, 1.003]$ the Poincaré map of the equation

$$z' = (1 + e^{i\varphi t}|z|^2)\bar{z},$$

admits a chaotic invariant set, which is semiconjugate to symbolic dynamics on two symbols.
Conclusions

- Strong expansion in a dynamical system does not necessarily mean that rigorous numerics of the system will not be helpful.
- Transferring information to topological level as soon as possible may be extremely helpful in solving problems, where other approaches fail because of rapid growth of error estimates.
- The presented methods may be applied not only to Poincaré maps in time periodic non-autonomous differential equations, but also to Poincaré maps in autonomous equations and $t$ translations.