Computer-Assisted Proof of the Stability of the Eight

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$N$-body Problem

The Newton’s equation for the movement of $N$ particles

$$m_i \ddot{q}_i = \sum_{j \neq i} \frac{Gm_im_j(q_j - q_i)}{\|q_i - q_j\|^3}$$

where

- $q_i \in \mathbb{R}^n$,
- $i, j = 1, \ldots, N$,
- $G = 6.6732 \times 10^{-11} \text{ m}^3/\text{s}^2\text{kg}$. 
The Newton’s equation for the movement of $N$ particles

\[ m_i \ddot{q}_i = \sum_{j \neq i} \frac{Gm_im_j(q_j - q_i)}{\|q_i - q_j\|^3} \]

In the following we assume that:

- $G = 1$
- $m_i = 1$
- $n = 2 \implies q_i = (x_i, y_i), \, \dot{q}_i = p_i = (\dot{x}_i, \dot{y}_i)$
The Eight - famous eight shaped orbit
Choreographies: starting from 2000 - C. Simó found numerically a lot of choreographies for various $N$, 

Gerver’s orbit - Super Eight
Definition

By a *choreography* we mean a collision-free solution of the N-body problem in which all masses move on the same curve with a constant phase shift.
Rigorous estimates for initial conditions
Computing the Eight monodromy matrix \( \frac{\partial \varphi}{\partial x} (Z, T) \).
Estimation of the eigenvalues of the matrix \( \frac{\partial \varphi}{\partial x} (Z, T) \).
Rigorous estimates of initial conditions

\[ G(x) = \sigma \varphi \left( \frac{T}{N}, x \right) \]

\[ x = (q_1, \dot{q}_1, q_2, \dot{q}_2, \ldots, q_N, \dot{q}_N) \]

\( \varphi(t, x) \) - flow generated by \( N \)-body equation.

\( \sigma(x) \) cyclical shift of the particles \( q_1 \rightarrow q_2 \rightarrow \cdots \rightarrow q_N \rightarrow q_1 \)

\[ F(x) = G(x) - x \]

\[ F(x_0) = 0 \iff G(x_0) = x_0 \iff \]
\[ \iff x_0 \text{ initial condition for some choreography.} \]
Interval Krawczyk Method

- $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a $C^1$ function,
- $X \subset \mathbb{R}^n$ is an interval set, $\bar{x} \in X$
- $C \in \mathbb{R}^{n \times n}$ is a linear isomorphism.

**Krawczyk operator**

$$K(\bar{x}, X, F) := \bar{x} - CF(\bar{x}) + (Id - C[DF(X)])(X - \bar{x});$$

**Krawczyk Theorem**

With the assumptions introduced above, the following holds:

1. If $x^* \in X$ and $F(x^*) = 0$, then $x^* \in K(\bar{x}, X, F)$.
2. If $K(\bar{x}, X, F) \subset intX$, then there exists a unique $x^* \in X$ such that $F(x^*) = 0$. 
In the case of Eight $\frac{\partial \varphi}{\partial x}(x_0, T)$ is an $8 \times 8$ matrix.

At least 4 eigenvalues are equal 1 (they correspond to the first integrals of the $N$-body equation).

We will show that remaining "relevant" eigenvalues are on the unit circle.

Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ denote "relevant" eigenvalues.
Non-rigorous estimations of eigenvalues (C. Simó):
\[ \lambda_{1,2} \approx 0.99859998 \pm 0.05289683i, \]
\[ \lambda_{3,4} \approx -0.29759667 \pm 0.95469169i, \]
If $\lambda$ is eigenvalue of simplectic matrix $A$ then also $\bar{\lambda}$, $\lambda^{-1}$, $\bar{\lambda}^{-1}$ are eigenvalues of $A$.

(S1) some of $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$ are real and $\lambda_1 \lambda_2 = 1$, $\lambda_3 \lambda_4 = 1$, 
Possible cases

(S2) \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are not real and \( \lambda_1 = \lambda_2^{-1} = \bar{\lambda}_3 = \bar{\lambda}_4^{-1} \),
(S3) \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) are different, not real and \( \lambda_1 = \lambda_2^{-1} = \bar{\lambda}_2 \),
\( \lambda_3 = \lambda_4^{-1} = \bar{\lambda}_4 \),
In the all above situations the characteristic polynomial is in the form:

$$(\lambda - 1)^4(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$$

$$= \lambda^8 - (T_1 + T_2 + 4)\lambda^7 + (T_1 T_2 + 4(T_1 + T_2) + 8)\lambda^6 + \ldots$$

where $T_1 = \lambda_1 + \lambda_2$, $T_2 = \lambda_3 + \lambda_4$. 
For any $8 \times 8$ matrix $A = (a_{ij})$ we have

$$\det(A - \lambda I) = \lambda^8 - \alpha \lambda^7 + \beta \lambda^6 + \ldots$$

(1)

where

$$\alpha = \text{trace } (A) = \sum_{i=1}^{8} a_{ii},$$

$$\beta = \sum_{1 \leq i < j \leq 8} (a_{ij}a_{jj} - a_{ij}a_{ji}).$$
From equations

\[ \text{det}(A - \lambda I) = \lambda^8 - \alpha \lambda^7 + \beta \lambda^6 + \ldots \]

\[ = \lambda^8 - (T_1 + T_2 + 4) \lambda^7 + (T_1 T_2 + 4(T_1 + T_2) + 8) \lambda^6 + \ldots \]

we get that \( T_1, T_2 \) are solution to:

\[ T^2 - (\alpha - 4) T + \beta - 4\alpha + 8 = 0. \]
Main Theorem

Theorem

Let $A$ be symplectic matrix having at least four eigenvalues equal to 1. Let $T_1$, $T_2$ be solutions of the equation

$$T^2 - (\alpha - 4)T + \beta - 4\alpha + 8 = 0.$$ 

If

$$\Delta = (\alpha - 4)^2 - 4(\beta - 4\alpha + 8) > 0,$$

$$|T_1| < 2, |T_2| < 2,$$

then all eigenvalues of matrix $A$ belong to unit circle.
For well chosen set $X$ and $\bar{x} \in X$ we:

- check that $Z := K(\bar{x}, X, F) \subset intX$
  (computation of $F(\bar{x}), [DF(X)]$)
- compute $A = \frac{\partial \phi}{\partial x}(Z, T)$
- check that
  - $\Delta = (\alpha - 4)^2 - 4(\beta - 4\alpha + 8) > 0$
  - $|T_1| < 2, |T_2| < 2$
We use:

- CAPD package: $C^n$ Lohner algorithm, set representation,
- MPFR package: multi-precision floating points numbers with correct rounding,
- MPFR++ - C++ envelope class for MPFR.
Rigorous Computations

<table>
<thead>
<tr>
<th>Set size $2 \cdot 10^{-9}$</th>
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</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>double</td>
</tr>
<tr>
<td>MP 53</td>
</tr>
<tr>
<td>MP 100</td>
</tr>
<tr>
<td>F(x₀)</td>
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<tr>
<td>F(X)</td>
</tr>
<tr>
<td>K(x₀,X,F)</td>
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<tr>
<td>time :</td>
</tr>
</tbody>
</table>

**Table:** Comparison of results for various precision.
Rigorous Computations

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<table>
<thead>
<tr>
<th></th>
<th>double</th>
<th>MP 56</th>
<th>MP 100</th>
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<tbody>
<tr>
<td>$F(x_0)$</td>
<td>$4.584e-12$</td>
<td>$5.273E-13$</td>
<td>$3.013e-26$</td>
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<tr>
<td>$\text{diam } F(X)$</td>
<td>$1.502e-06$</td>
<td>$1.423E-6$</td>
<td>$1.412E-6$</td>
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<tr>
<td>$K(x_0,X,F)$</td>
<td>$1.352e-10$</td>
<td>$1.556E-11$</td>
<td>$1.291E-16$</td>
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<tr>
<td>time:</td>
<td>$32.9$ sec</td>
<td>$922$ sec</td>
<td>$999$ sec</td>
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Rigorous Computations

<table>
<thead>
<tr>
<th>Set size $2 \cdot 10^{-12}$</th>
<th>MP 100</th>
<th>MP 200</th>
<th>MP 400</th>
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<tbody>
<tr>
<td>$F(x_0)$</td>
<td>3.013e-26</td>
<td>6.451E-31</td>
<td>6.451E-31</td>
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<tr>
<td>diam $F(X)$</td>
<td>1.412E-6</td>
<td>1.412E-6</td>
<td>1.412E-6</td>
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<tr>
<td>$K(x_0,X,F)$</td>
<td>1.291E-16</td>
<td>1.291E-16</td>
<td>1.291E-16</td>
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<tr>
<td>time:</td>
<td>999 sec</td>
<td>1041 sec</td>
<td>1150 sec</td>
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**Table:** Comparison of results for various precision.
Theorem

All eigenvalues of the Eight monodromy matrix belong to unit circle.

<table>
<thead>
<tr>
<th>Initial conditions</th>
<th></th>
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<tr>
<td>set size Z</td>
<td>1.29e-16</td>
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<td>precision</td>
<td>100 mantissa bits</td>
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<table>
<thead>
<tr>
<th>Results of computation</th>
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<tbody>
<tr>
<td>$\Delta$</td>
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<tr>
<td>$6.720_{458965}^{547070}$</td>
</tr>
<tr>
<td>$T_1$</td>
</tr>
<tr>
<td>$1.997_{195667}^{204261}$</td>
</tr>
<tr>
<td>$T_2$</td>
</tr>
<tr>
<td>$-0.5951_{87631}^{89038}$</td>
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we can verify numeric simulations

having rigorous estimates for initial conditions we can proof additional properties of the given orbit: stability, symmetries, . . .

good estimates needed:
- multiprecision interval arithmetics
- set representation: Taylor Models

full stability $\rightarrow$ normal forms calculations