Rigorous Global Optimization of Impulsive Space Trajectories

P. Di Lizia, R. Armellin, M. Lavagna  
*Politecnico di Milano*

K. Makino, M. Berz  
*Michigan State University*

Fourth International Workshop on Taylor Methods  
Boca Raton, December 16 – 19, 2006
Motivation

- Space activities are expensive:
  
  Ariane 5 launch cost: \(200 \text{ M$\div}\)
  
  Allowed Spacecraft Mass: \(10000 \text{ kg}\)
  
  Cost per kilogram: \(20000 \text{ \$/kg}\)
  
  more expensive than gold (as expensive as saffron)

- Propellant represents the main contribution to s/c mass:
  
  Propellant is on average 40% of spacecraft mass
  
  we want to reduce the required propellant

- The goal of the trajectory design is to find the best solution in terms of propellant consumption while still achieving the mission goals
Outline

- Dynamical Model
- Patched-Conics Approximation
- Two-Impulse Transfers
  - Ephemerides Evaluation
  - Lambert’s Problem Solution
- Differential Algebra Based Global Optimization
- Rigorous Global Optimization with COSY-GO
Dynamical Model: 2-Body Problem

• The 2-Body Problem considers two point masses in mutual orbit about each other

The relative motion of the two masses is governed by:

$$\ddot{\vec{r}} = -\frac{k}{r^3} \vec{r}$$

E.g.

- $m_1 \rightarrow$ Sun
- $m_2 \rightarrow$ Spacecraft

Analytical solutions exist for the 2-Body Problem: \textit{Conic Arcs}

- $\vec{r} = \vec{r}(\theta)$ $\rightarrow$ explicit
- $t = t(\theta)$ $\rightarrow$ implicit \textit{(Kepler’s equation)}
Patched-Conics Approximation

- The whole interplanetary transfer is divided in several arcs
- Each arc is the solution of a 2-Body Problem considering the spacecraft and only one other planet at a time

E.g.: 2-impulse Earth-Mars transfer → 3 conic arcs

Earth escape  Heliocentric phase  Mars capture
2-Impulse Planet-to-Planet Transfer

- 2-impulse Earth-Mars transfer has been selected as first benchmark problem
  - Applied for preliminary design of Earth-Mars interplanetary transfers
  - Objective function characterized by several comparable local minima

- Future benchmark problems
  - Multiple Gravity Assist interplanetary transfers
    E.g.: Cassini-Huygens (11 conic arcs)
Optimization Problem

- The optimization variables are the time of departure $t_0$ and the time of flight $t_{tof}$
- The positions of the starting and arrival planets are computed through the ephemerides evaluation:
  $$(r_E, v_E) = \text{eph}(t_0, \text{Earth}) \quad \text{and} \quad (r_M, v_M) = \text{eph}(t_0 + t_{tof}, \text{Mars})$$
- The starting velocity $v_1$ and the final one $v_2$ are computed by solving the Lambert’s problem
- Objective function:
  $$\Delta V = \Delta V_1 + \Delta V_2$$
- Constraint:
  $$\Delta V_1 < \Delta V_{1,\text{max}}$$
Orbital parameters

- The orbital parameters are: \((a, e, i, \Omega, \omega, \theta)\)

- The position and the velocity \((r, v)\) in cartesian coordinates are obtained from the orbital parameters by simple algebraic relations.
Ephemerides Evaluation

- Polynomial interpolations of accurate planetary ephemerides (JPL-Horizon) are used for the preliminary phase of the space trajectory design.

- Given an epoch and a celestial body, its orbital parameters \((a, e, i, \Omega, \omega, M)\) can be analytically evaluated.

- The nonlinear equation \(M = E - e \sin E\) (Kepler’s Eq) is solved for the eccentric anomaly \(E\).

- The relation \(\tan \frac{\theta}{2} = \sqrt{\frac{1 + e}{1 - e}} \tan \frac{E}{2}\) delivers \(\theta\).

- The position and the velocity \((r, v)\) of the celestial body in inertial frame reference frame are computed.

We have to solve an implicit equation: Kepler’s equation.
Lambert’s Problem

Given:
• initial position \( r_1 \)
• final position \( r_2 \)
• time of flight \( t_{tof} \)

Find the initial velocity, \( v_1 \), the spacecraft must have to reach \( r_2 \) in \( t_{tof} \)

• The solution of the BVP exploits the analytical solution of the 2-body problem
• Given \( r_1 \), \( r_2 \) and \( t_{tof} \) there exists only one conic arc connecting the two points in the given time
Lambert’s Problem

• Several algorithms have been developed for the identification and characterization of the resulting conic arc

• We used an algorithm developed by Battin (1960)

• A nonlinear equation must be solved (Lagrange’s equation for the time of flight):

\[ f(x) = \log(A(x)) - \log(t_{tof}) = 0 \]

in which \( A(x) = a(x)^{3/2}((\alpha(x) - \sin(\alpha(x))) - (\beta(x))) \),

\[ \beta(x) = 2 \arcsin \left( \frac{s - c}{2a(x)} \right) \]

\[ a(x) = \frac{s}{2(1 - x^2)} \]

• The value of \( s \) and \( c \) depend on \( r_1 \) and \( r_2 \), so the nonlinear equation depends both on \( t_0 \) and \( t_{tof} \)
DA Based Global Optimizer (1/2)

DA based global optimization algorithm:

- Subdivide the search space in subintervals $t_{tof}$
- Suitably initialize the value of $\Delta V_{opt}$

For each subinterval $\vec{X}$:

- Initialize $t_0$ and $t_{tof}$ as DA variables and compute a Taylor expansion of the objective function $\Delta V$ and the constraint $\Delta V_1$ on $\vec{X}$
- Bound the value of $\Delta V_1$ on $\vec{X}$
  
  IF $\min \Delta V_1 > \Delta V_{1,max}$ → discard $\vec{X}$

- Bound the value of $\Delta V$ on $\vec{X}$
  
  IF $\min \Delta V > \Delta V_{opt}$ → discard $\vec{X}$
DA Based Global Optimizer (2/2)

- Build and invert the map of the objective function gradient:

\[
\begin{pmatrix}
\nabla_{t_0} \Delta V \\
\nabla_{t_{tof}} \Delta V
\end{pmatrix} = \mathcal{M}
\begin{pmatrix}
t_0 \\
t_{tof}
\end{pmatrix} = \mathcal{M}^{-1}
\begin{pmatrix}
\nabla_{t_0} \Delta V \\
\nabla_{t_{tof}} \Delta V
\end{pmatrix}
\]

- Localize the zero-gradient point \( \vec{x}^* = (t_0^*, t_{tof}^*) \)

**IF** \( \vec{x}^* \notin \vec{X} \) \quad **discard** \( \vec{X} \)

- Evaluate \( \Delta V^* = \Delta V(\vec{x}^*) \)

**IF** \( \Delta V^* < \Delta V_{opt} \) \quad **update** \( \Delta V_{opt} \), and store \( \vec{x}^* \) and \( \vec{X} \)

- If necessary, a more accurate identification of the actual optimum \( \vec{x}^* \) can be finally achieved using a higher order DA computation on the last stored subinterval \( \vec{X} \)
DA Solution of Parametric Implicit eqs

- Search the solution of \( f(x, p) = 0 \) for \( p \) belonging to \( p \in [p_l, p_u] \)
- Use classical methods (e.g., Newton) to compute \( x^0 \) solution of \( f(x, p^0) = 0 \)
- Initialize \([x] = x^0 + \Delta x\) and \([p] = p^0 + \Delta p\) as DA variables and expand \( \Delta f = M(\Delta x, \Delta p) \)
- Build the following map and invert it:

\[
\begin{pmatrix}
\Delta f \\
\Delta p
\end{pmatrix} =
\begin{pmatrix}
[M] \\
[I_p]
\end{pmatrix}
\begin{pmatrix}
\Delta x \\
\Delta p
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
\Delta x \\
\Delta p
\end{pmatrix} =
\begin{pmatrix}
[M]^{-1} \\
[I_p]
\end{pmatrix}
\begin{pmatrix}
\Delta f \\
\Delta p
\end{pmatrix}
\]

- Force \( \Delta f = 0 \) so obtaining the Taylor expansion of the solution w.r.t. the parameter: \( \Delta x = \Delta x(\Delta p) \)
Example: Mars Ephemerides

Epoch interval: 40 days

Errors on position, (a), and velocity, (b), between the DA and the point-wise evaluation of Mars ephemerides

Errors drastically decrease when the order of the Taylor series increases
Example: Objective Function

- The DA evaluation of the planetary ephemerides and the Lambert’s problem solution enables the Taylor expansion of the objective function.

  **Taylor representation of the objective function**
  
  **Taylor representation error w.r.t. point-wise evaluation**

Box width: 40 days
Earth-Mars Direct Transfer

Search space: $[1000, 6000] \times [100, 600]$  
Maximum departure impulse: $\Delta V_1 < 5 \text{ km/s}$  
Platform: Pentium IV 3.06 GHz laptop

Objective function overview
Earth-Mars Direct Transfer

Search space: \([1000, 6000] \times [100, 600]\)

Maximum departure impulse: \(\Delta V_1 < 5 \text{ km/s}\)

Platform: Pentium IV 3.06 GHz laptop

Solution 1:
- 10-day boxes + 5th order
- Pruning + Global Opt: 59.98 s
- \(\Delta V^* = 5.6673 \text{ km/s}\)
- \(x^* = [3573.188, 324.047]\)

Solution 2:
- 100-day boxes + 5th order
- Pruning + Global Opt: 0.55 s
- \(\Delta V^* = 5.6676 \text{ km/s}\)
- \(x^* = [3573.530, 323.371]\)
Suppose to have the \((n + 1)\) differentiable function \(f\) over the domain \(D = [-1, 1]\) and its \(n\)-th order Taylor model \(P(x) + I\) so that
\[
f(x) \in P(x) + I \quad \text{for all } x \in D
\]

Consider the enclosure \(R\) of \(P(x) + I\) over \(D\) and suppose \(P'(x) > d > 0\) on \(D\) with \(P(0) = 0\)

Find the Taylor Model \(C(y) + J\) on \(R\) so that any solution of the problem \(f(x) = y\) lies in \(C(y) + J\)

Algorithm:

First compute \(C(y)\), the \(n\)-th order polynomial inversion of \(P(x)\), so that
\[
P(C(y)) =_n y
\]

Using Taylor model computation, obtain \(P(C(y)) \in y + \tilde{J}\) where \(\tilde{J}\) includes the terms of order exceeding \(n\) in \(P(C(y))\), and thus scales with at least order \(n + 1\)
• Use the consequences of small correction $\Delta x$ to $C(y)$ to find the rigorous reminder $J$ for $C(y)$ so that all the solutions of $f(x) = y$ lie in $C(y) + J$. According to the mean value theorem:

$$f(C(y) + \Delta x) - y \in P(C(y) + \Delta x) - y + I$$

$$= P(C(y)) + \Delta x \cdot P'(\xi) - y + I$$

$$\subset y + \tilde{J} + \Delta x \cdot P'(\xi) - y + I$$

$$= \Delta x \cdot P'(\xi) + I + \tilde{J}$$

for suitable $\xi \in [C(y), C(y) + \Delta x]$

• Since $P'$ is bounded below by $d$, the set $\Delta x \cdot P'(\xi) + I + \tilde{J}$ will never contain the zero except for the interval

$$J = -\frac{I + \tilde{J}}{d}$$

which is the desired interval
Verified GO of Earth-Mars Transfer

- The previous verified solver of implicit equations enabled the Taylor Model evaluation of the objective function.
- COSY-GO has been applied for the global optimization of the impulsive Earth-Mars transfer.

- Number of steps: 216911
- Computation time: 4954.39 s
- Enclosure of the minimum: [5.6673264, 5.6673272] km/s
- Enclosure of the solution:
  \[ t_0 \in [3573.176, 3573.212] \]
  \[ t_{tof} \in [324.034, 324.088] \]
Conclusions and Future Work

Conclusions:

• DA and TM global optimizers are promising and efficient tools for the global optimization of a space mission
• Efficient management of discontinuities is needed for TM global optimization with COSY-GO

Future Work:

• Extend the models to Multiple Gravity Assist (MGA) interplanetary transfers and Deep Space Maneuvers
• Exploit the embedded domain box feature to prune the search space in MGA transfers
• Enable a dynamic selection of the box size based on a suitable definition of the trust region of a Taylor expansion in the DA based GO algorithm
Rigorous Global Optimization of Impulsive Space Trajectories

P. Di Lizia, R. Armellin, M. Lavagna
Politecnico di Milano

K. Makino, M. Berz
Michigan State University